1. Prove the **Inclusion–Exclusion Identity** for two events:

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

2. **Expanded definition of RV:** Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a measurable space \((E, \mathcal{E})\), an \((\mathcal{E}/\mathcal{F})\)-**measurable random variable** is a measurable function \(X: \Omega \rightarrow E\).

Consider the function \(f: X \rightarrow Y\), where \(Y\) is a countable space. Choose the \(\sigma\)-algebra \(Y = 2^Y\) and choose any \(\sigma\)-algebra \(X\) on \(X\). Show that \(f\) is \((Y/X)\)-measurable if and only if \(f^{-1}(i) \in X\).

3. We will flip three fair coins, and use \(\Omega = \{H, T\}^3\) as our probability space. We define \(X\) as the total number of heads on the three flips, and define \(\mathcal{F}_1\) as the information gained after one flip.

   (a) Show that \(E[X|\mathcal{F}_1]: \Omega \rightarrow \mathbb{R}\) only takes two values, and determine the sets on which they are constant.

   (b) Compute these two values using the theorem about linearity of expectation, by writing everything out explicitly.

4. Consider the same filtration as for the previous exercise: \(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\), generated by the information gained after one, two and three flips, respectively. Let \(X\) be the random variable that counts whether or not an odd number of heads have been flipped, i.e. \(X(\omega) = 1\) if \(\omega\) has an odd number of \(H\)s, and 0 otherwise.

   (a) Give an exact description of the random variable \(E[X|\mathcal{F}_2](\omega)\).

   (b) Compute \(E[E[X|\mathcal{F}_2]]\) by hand and check that it matches the Law of Total Expectation.

5. **Definition:** If \(X: \Omega \rightarrow I\) is a discrete RV, then the **probability generating function (pgf)** of \(X\) is the function \(\phi_X(t) := E[t^X] = \sum_{i \in I} t^i \mathbb{P}(X = i)\). You can use the fact that the pgf of a sum of independent RVs is the product of their pgfs, as with the moment generating function.

A Bernoulli trial is a random variable \(X\) that takes values zero and one, and \(\mathbb{P}(X = 1) = p\) (implying, of course, that \(\mathbb{P}(X = 0) = 1 - p\)). This can be thought of as an experiment that has probability \(p\) of success and \(1 - p\) of failure, e.g., an unfair coin. The binomial distribution \(B(n, k)\) is defined as the probability of having \(k\) successes if we consider \(n\) independent Bernoulli trials. Compute the probability generating function of a Bernoulli trial, and use this to compute (i.e., identify the coefficients in the power series) the probability distribution \(B(n, k)\).
6. If $X \sim U(0,1)$, show that, for any $0 \leq a < b \leq 1$,
$$
P(X \in [a,b]) = P(X \in [a,b)) = P(X \in (a,b)) = P(X \in (a,b]) = b - a.
$$

7. Let $X, Y, Z, W$ be independent $U(0,1)$ random variables. Use a Monte Carlo method to compute $E[XY^2 + e^Z \cos(W)]$. How much computation should you do to be confident in your answer to three decimal places? Turn in your code or pseudo-code along with your answer.

8. We say that $T$ is an exponential random variable with parameter $\mu$ if $T > 0$ with probability 1 and
$$
P(T > t) = e^{-\mu t}, \text{ or } P(T \leq t) = 1 - e^{-\mu t}.
$$
First, write a function that allows you to sample $T$ using a stream of $U(0,1)$ variables. Next, set $\mu = 2$ and compute $E[T^p]$ for $p = 1, 2, 3, 4$. Do you see a pattern?