1. Consider the Markov chain with transition matrix

\[ P = \begin{pmatrix} 1 - p_{12} & p_{12} \\ p_{21} & 1 - p_{21} \end{pmatrix}. \]

Compute the invariant distribution for \( P \), and show that \( \pi_1 \) is monotone decreasing in \( p_{12} \) and monotone increasing in \( p_{21} \).

Now consider a three-state “chain” with matrix

\[ P = \begin{pmatrix} 1 - p_{12} & p_{12} & 0 \\ p_{21} & 1 - p_{21} - p_{23} & p_{23} \\ 0 & p_{32} & 1 - p_{32} \end{pmatrix}. \]

Compute the invariant distribution for this Markov chain. Make a conjecture for monotonicity that is analogous for the 2 \( \times \) 2 case. Does this still hold here?

2. Consider the two state Markov chain with state \( \{1, 2\} \) and with transition probabilities \( p_{12} = p, p_{21} = q \). Let us assume that \( 0 < q < 1 \) is fixed but we can choose \( p \in [0, 1] \) however we like. Moreover, assume that there is a payoff of \( r > 0 \) every time we visit state 2, and a cost of \( c(p) \) every time we visit state 1. Then:

(a) Compute the long-term profit per time step as a function of \( p \). Describe a method to find the optimal choice of \( p \) to maximize profits. Does such an optimal choice always exist?

(b) Assume that the cost function is linear, i.e. that \( c(p) = \alpha p \) for some \( \alpha > 0 \). Show that the optimal choice of \( p \) will lead to a profit-per-step of

\[ \left( \frac{r - \alpha q}{1 + q} \right)^+, \]

where we denote \( x^+ := \max\{x, 0\} \).

(c) Assume that the cost function is constant, i.e. \( c(p) = c \) for all \( p \). What is the optimal choice of \( p \)? What is the optimal profit?

3. Here we will prove some useful facts for the proof of the Ergodic Theorem below. Consider any positive recurrent Markov chain \( P \) on a state space \( I \).

- For any \( \epsilon > 0 \), show that there is a finite subset \( J \subseteq I \) such that \( \sum_{i \in J} \pi_i > 1 - \epsilon \).
- Use the fact that \( \pi \) is a distribution to show that, for any subset \( J \subseteq I \),

\[ \sum_{i \notin J} |U_i(n) - \pi_i| \leq 2 \sum_{i \notin J} \pi_i + \sum_{i \in J} |U_i(n) - \pi_i|. \]
Hint. First justify, and then use, the facts that \( \sum_{i \in J} \pi_i = 1 - \sum_{i \notin J} \pi_i \) and \( \sum_{i \in J} U_i(n) = 1 - \sum_{i \notin J} U_i(n) \).

**Theorem** (Ergodic Theorem, for reference, not to prove) If \( P \) is irreducible and positive recurrent with invariant measure \( \pi \), then for any bounded observable \( f: I \to \mathbb{R} \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \langle \pi, f \rangle, \text{ almost surely.}
\]

4. Recall the explosion time of a CTMC is \( \zeta := \lim_{n \to \infty} J_n \).

   (a) Define a CTMC on a state space \( I \) and a state \( i \in I \) such that \( \mathbb{P}_i(\zeta < \infty) = 1/2 \).

   (b) Generalize the example in part (a) from \( 1/2 \) to an arbitrary number \( a \in (0, 1) \).

   (c) Is it possible to define a CTMC and \( i \) with \( \mathbb{P}_i(\zeta < \infty) = a \in (0, 1) \) and \( \mathbb{P}_j(\zeta < \infty) < 1 \) for all \( j \)? Why or why not?

5. For this problem, assume \( T_1, T_2, \ldots, T_n \) are independent RVs and \( T_k \sim \text{Expon}(\lambda_k) \).

   (a) Let \( T := \min_{i=1,\ldots,n} T_i \). Show that \( T \) is exponentially distributed and compute its rate.

   (b) Compute the statistics of \( T_1 + T_2 \) (mean, variance, whole distribution).

   (c) Compute the characteristic function of \( T_k \), \( \varphi(s) = \mathbb{E}[e^{sT_k}] \).