461 Midterm 1 Solutions

(1) (15 pts) Fifteen distinct balls are to be divided randomly among players A, B, and C, with each one getting five balls. If 10 balls are White and 5 are Orange, then find the probability that player A gets 2 White, player B gets 3 White, and player C gets 5 White.

Without ordering

$$\frac{\binom{10}{2,3,5}\binom{5}{3,2,0}}{\binom{15}{5,5,5}} = \frac{\frac{10!}{2!3!5!}\frac{5!}{3!2!}}{\frac{15!}{5!5!5!}}$$

or

$$\frac{\binom{10}{2}\binom{5}{3}\binom{8}{3}}{\binom{15}{5}\binom{10}{5}\binom{5}{5}} = \frac{\frac{10!}{2!3!2!3!}}{\frac{15!}{5!5!5!}}$$

With ordering

$$\underbrace{\_\ \underset{\text{W}}{\text{X}}\ \text{X}\ \_\ \_}_{A\ 5}\quad \underbrace{\_\ \_\ \text{X}\ \text{X}\ \text{X}}_{B\ 5}\quad \underbrace{\text{X}\ \text{X}\ \text{X}\ \text{X}\ \text{X}}_{C\ 5}$$

$$\frac{\overbrace{\binom{5}{2}\binom{5}{3}\binom{5}{5}}^{\text{pos'ns of W balls}}\ \overset{W}{10!}\ \overset{O}{5!}}{15!}$$

(2) (15 points) A bridge hand of 13 cards is randomly dealt from a standard deck of 52 cards. What is the probability that the hand is void in (i.e., is missing) at least one suit?

S : void in spades C: void in clubs

D : — diamonds H: — hearts

I-E Id.

$$\mathbb{P}(S \cup C \cup D \cup H) = \mathbb{P}(S) + \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(H)$$
$$- \mathbb{P}(S \cap C) - \cdots$$
$$+ \mathbb{P}(S \cap C \cap D) + \cdots$$
$$- \cancel{\mathbb{P}(S \cap C \cap D \cap H)} \to 0$$
$$= \binom{4}{1} \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3} \left(\frac{\binom{13}{13}}{\binom{52}{13}} \right)$$

Extra credit: What is wrong with the reasoning that leads to the incorrect answer $\binom{4}{1}\binom{39}{13}/\binom{52}{13}$?

overcounts being void in > 1 suits.

(3) (15 pts) (a) (5 pts) What is the definition of conditional probability?

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

(b) (10 pts) A parallel system works whenever at least one of its components works. Consider a parallel system with ~~n~~ ≥ 2 components and suppose that each component independently works with probability 2/3. Find the probability that the first two components are working conditioned on the system working.

$$\mathbb{P}(\text{first two work} \mid \text{system works}) = \mathbb{P}(F|S)$$

$$= \frac{\mathbb{P}(F \cap S)}{\mathbb{P}(S)}$$

$$= \frac{\mathbb{P}(F)}{1 - \mathbb{P}(S^c)}$$

$$= \frac{\left(\frac{2}{3}\right)^2}{1 - \left(\frac{1}{3}\right)^n}$$

(4) (~~15~~ 10 pts) A blood test is 99% effective in detecting a disease given that the disease is actually present. The test also has a 1% chance of being positive when the disease is not actually present. If 2% of the population actually has the disease, then (a) (10 pts) what is the probability that a person actually has the disease given that their test result is positive?

$$\mathbb{P}(\text{Dis.}\mid \text{Pos.}) = \frac{\mathbb{P}(D\cap P)}{\mathbb{P}(P)}$$

$$= \frac{\mathbb{P}(P\mid D)\,\mathbb{P}(D)}{\underbrace{\mathbb{P}(P\mid D)\,\mathbb{P}(D)}_{\mathbb{P}(D\cap P)} + \underbrace{\mathbb{P}(P\mid D^c)\,\mathbb{P}(D^c)}_{\mathbb{P}(D^c\cap P)}}$$

$$= \frac{.99\cdot .02}{.99\cdot .02 + .01\cdot .98}$$

$$\approx \frac{.02}{.02+.01} = \frac{2}{3}$$

(b) (5 pts) Heuristically, what should this probability approximately be?

Test 100 people:

≈ 2 true positives

& ≈ 1 false positives

So $\mathbb{P}(D\mid \text{Pos.}) \cong \frac{2}{2+1} = \frac{2}{3}$

(5) (20 pts) Let X be a random variable with distribution function F given by

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{4}, & 0 \le x < 1, \\ \frac{1}{2}, & 1 \le x < 2, \\ \frac{x+6}{12}, & 2 \le x < 3, \\ 1, & 3 \le x. \end{cases}$$

(a) Is this a probability distribution function or a cumulative distribution function? (5 pts)

Cumulative

Compute (3 pts each)
(b) $\mathbb{P}(X = 2)$, (c) $\mathbb{P}(1 \le X < 3)$, (d) $\mathbb{P}(X > \frac{3}{2})$,
(e) $\mathbb{P}(2 < X \le 7)$, and (f) $\mathbb{P}(X < 2)$.

(b) $\mathbb{P}(X=2) = F(2) - F(2-) = \frac{8}{12} - \frac{6}{12} = \frac{1}{6}$

(c) $\mathbb{P}(1 \le X < 3) = F(3-) - F(1-) = \frac{9}{12} - \frac{3}{12} = \frac{1}{2}$

(d) $\mathbb{P}(X > 3/2) = 1 - F(3/2) = 1 - \frac{1}{2} = \frac{1}{2}$

(e) $\mathbb{P}(2 < X \le 7) = F(7) - F(2) = 1 - \frac{8}{12} = \frac{1}{3}$

(f) $\mathbb{P}(X < 2) = F(2-) = \frac{1}{2}$

Binomial $(n, \frac{1}{6})$

(6) (20 pts) Consider n rolls of a fair die, and let the random variable X be the number of times that the top face is 6.

(a) (10 pts) Compute the probability mass function of X.

$$p(i=0) = \binom{n}{0}\left(\frac{5}{6}\right)^n$$

$$p(1) = \binom{n}{1}(5/6)^{n-1}(1/6)$$

$$p(2) = \binom{n}{2}(5/6)^{n-2}(1/6)^2$$

$$\vdots$$

$$p(n) = \binom{n}{n}(1/6)^n$$

So $p(i) = \binom{n}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{n-i}$

(b) (5 pts) Compute the expected value of X.

$$\mathbb{E}(X) = \sum_{k=0}^{n} k\,p(k)$$

$$= \sum_{k=\cancel{0}1}^{n} k\binom{n}{k}\left(\frac{5}{6}\right)^{n-k}\left(\frac{1}{6}\right)^k$$

$$= \frac{n}{6}$$

because $\mathbb{E}(\mathrm{Bin}(n,p)) = np$

(c) (5 pts) If the variance of X is σ^2, then what is $\underbrace{\mathbb{E}[(X-\mathbb{E}(X))^2]}_{\text{this is } 1^{st} \text{ defn of } Var(X).}$? $= \sigma^2$