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CLT: X_i i.i.d. with $\mathbb{E}[X_i]=\mu$ & $Var(X_i)=\sigma^2$

$$\Longrightarrow \quad Y_n := \frac{X_1+X_2+\dots+X_n-n\mu}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{} Z\sim\mathcal{N}(0,1),$$

or $\mathbb{P}(Y_n \le a) \longrightarrow \Phi(a)$ as $n\to\infty$.

Proof (of special case: $\mu=0$, $\sigma^2=1$, $Y_n=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$

Use Fact: RVs $Y_n \to Z$ iff their mgfs converge: $\boxed{M_{Y_n}(t) \longrightarrow M_Z(t) \text{ as } n\to\infty.}$

Now
$$\begin{aligned}
M_{Y_n}(t) &= M_{\frac{1}{\sqrt{n}}X_1+\dots+\frac{1}{\sqrt{n}}X_n}(t) \\
&= M_{\frac{X_1}{\sqrt{n}}}(t)\cdot\ldots\cdot M_{\frac{X_n}{\sqrt{n}}}(t) \\
&= \mathbb{E}\left[e^{\frac{tX_1}{\sqrt{n}}}\right]\cdot\ldots\cdot\mathbb{E}\left[e^{\frac{tX_n}{\sqrt{n}}}\right] \\
&= M_{X_1}\left(\tfrac{t}{\sqrt{n}}\right)\cdot\ldots\cdot M_{X_n}\left(\tfrac{t}{\sqrt{n}}\right) \\
&= \left(M\left(\tfrac{t}{\sqrt{n}}\right)\right)^n, \text{ where}
\end{aligned}$$

$M(t) := M_{X_1}(t) = M_{X_i}(t)$, identical mgfs.

Goal:

$$M_{Y_n}(t) = \left(M\left(\tfrac{t}{\sqrt{n}}\right)\right)^n \xrightarrow[n\to\infty]{} M_Z(t) = e^{t^2/2}$$

Take logs: $\quad n\log M\left(\frac{t}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{?} \frac{t^2}{2}.$

$M(0) = \mathbb{E}[e^0] = 1.$

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First try: $\lim_{n\to\infty} n \cdot \log M(\frac{t}{\sqrt{n}})$ "=" $\infty \cdot \log M(0) = \infty \cdot 0$

Indeterminate.

L'Hôpital: (and replace n by x)

chain rule

$\frac{0}{0}$ form.

$$\lim_{x\to\infty} \frac{\log M(\frac{t}{\sqrt{x}})}{x^{-1}} = \lim_{x\to\infty} \frac{\frac{M'(\frac{t}{\sqrt{x}})}{M(\frac{t}{\sqrt{x}})} \cdot \frac{-t}{2x^{3/2}}}{-x^{-2}}$$

$$= \lim \frac{\frac{M'(\ )}{M(\ )} \cdot t}{2x^{-1/2}} \text{ "=" } \frac{\frac{\mu}{1} \cdot t}{0}$$, bad at $t=0$, & $\mu = 0$

L'Hôpital again:

derivs w.r.t. x

arguments are $\frac{t}{\sqrt{x}} \to 0$ as $x \to \infty$

$$\lim_{x\to\infty} \frac{\log M(\ )}{x^{-1}} = \lim_{x\to\infty} \frac{\frac{M'(\ )}{M(\ )} t}{2x^{-1/2}} = \text{quotient rule on top}$$

$$= \lim \frac{\frac{MM'' - M'M'}{M^2}\left(\frac{-t}{2\cancel{x^{3/2}}}\right) \cdot t}{-\cancel{x^{-3/2}}}$$

now cancels

$$= \frac{t^2}{2} \cdot \lim_{x\to\infty} \frac{M(\frac{t}{\sqrt{x}})M''(\frac{t}{\sqrt{x}}) - (M'(\frac{t}{\sqrt{x}}))^2}{M(\frac{t}{\sqrt{x}})^2}$$

$$= \frac{t^2}{2} \cdot \frac{M(0)M''(0) - (M'(0))^2}{M(0)^2}$$

$\sigma = 1$

$\mu = 0$

$$= \frac{t^2}{2} \cdot \frac{1 \cdot \sigma - \mu \cdot \mu}{1^2}$$

$$= \frac{t^2}{2}. \quad \text{QED.}$$