Ramsey-Minimal Saturation Numbers for Matchings

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Abstract

Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{F}$ is a subgraph of $G + e$. Let $sat(n, \mathcal{F})$ denote the minimum number of edges in an $\mathcal{F}$-saturated graph of order n, which we refer to as the *saturation number* or *saturation function* of $\mathcal{F}$. If $\mathcal{F} = \{F\}$, then we instead say that G is *F-saturated* and write $\text{sat}(n, F)$.

For graphs $G, H_1, \ldots, H_k$, we write that $G \to (H_1, \ldots, H_k)$ if every k-coloring of $E(G)$ contains a monochromatic copy of H_i in color i for some i. A graph G is *$(H_1, \ldots, H_k)$-Ramsey-minimal* if $G \to (H_1, \ldots, H_k)$ but for any $e \in G$, $(G - e) \not\to (H_1, \ldots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_k)$-Ramsey-minimal graphs.

In this paper, motivated in part by a conjecture of Hanson and Toft [Edge-colored saturated graphs, *J. Graph Theory* **11** (1987), 191–196], we prove that

$$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k)$$

for $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, and we also characterize the saturated graphs of minimum size. The proof of this result uses a new technique, *iterated recoloring*, which takes advantage of the structure of H_i-saturated graphs to determine the saturation number of $\mathcal{R}_{\min}(H_1, \ldots, H_k)$.

Keywords: saturated graph, Ramsey-minimal graph, matching

1 Introduction

All graphs considered in this paper are simple, undirected and finite. For any undefined terminology or notation, please see [7]. Given an edge coloring ϕ of a graph G let G_ϕ denote the edge-colored graph obtained by applying ϕ to G, and let $G_\phi[i]$ denote the spanning subgraph of G_ϕ induced by all edges of color i. When the context is clear, we will simply write G and $G[i]$ in place of the more cumbersome G_ϕ and $G_\phi[i]$.

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Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{F}$ is a subgraph of $G+e$. If $\mathcal{F}=\{F\}$, then we say that G is *F-saturated*. The classical extremal function $\mathrm{ex}(n,\mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-saturated graph of order n.

In this paper, we are concerned with $\mathrm{sat}(n,F)$, the minimum number of edges in an $\mathcal{F}$-saturated graph of order n. We refer to $\mathrm{sat}(n,\mathcal{F})$ as the *saturation number* or *saturation function* of $\mathcal{F}$. This parameter was introduced by Erdős, Hajnal and Moon in [2], wherein they determined $sat(n,K_t)$ and characterized the unique saturated graphs of minimum size. Here "$\vee$" denotes the standard graph join.

Theorem 1. *If n and t are positive integers such that $n\geq t$, then*

$$sat(n,K_t)=\binom{t-2}{2}+(t-2)(n-t+2).$$

Furthermore, $K_{t-2}\vee\overline{K}_{n-t+2}$ is the unique K_t-saturated graph of order n with minimum size.

Subsequently, $\mathrm{sat}(n,\mathcal{F})$ has been determined for a number of families of graphs and hypergraphs. We refer the interested reader to the dynamic survey of Faudree, Faudree and Schmitt [3], which gives a thorough overview of the area.

For graphs $G,H_1,\ldots,H_k$, we write that $G\to(H_1,\ldots,H_k)$ if every k-coloring of $E(G)$ contains a monochromatic copy of H_i in color i for some i. The (classical) Ramsey number $r(H_1,\ldots,H_K)$ is the smallest positive integer n such that $K_n\to(H_1,\ldots,H_k)$. A graph G is *$(H_1,\ldots,H_k)$-Ramsey-minimal* if $G\to(H_1,\ldots,H_k)$ but for any $e\in G$, $(G-e)\not\to(H_1,\ldots,H_k)$. Let $\mathcal{R}_{\min}(H_1,\ldots,H_k)$ denote the family of $(H_1,\ldots,H_k)$-Ramsey-minimal graphs.

Here we are interested in the following general problem.

Problem 1. *Let $H_1,\ldots,H_k$ be graphs, each with at least one edge. Determine*

$$\mathrm{sat}(n,\mathcal{R}_{\min}(H_1,\ldots,H_k)).$$

It is straightforward to prove that $G\to(H_1,\ldots,H_k)$ if and only if G contains an $(H_1,\ldots,H_k)$-Ramsey-minimal subgraph. Hence Problem 1 is equivalent to finding the minimum size of a graph G of order n such that there is some k-edge-coloring of G that contains no copy of H_i in color i for any i, yet for any $e\in\overline{G}$ every k-edge-coloring of $G+e$ contains a monochromatic copy of H_i in color i for some i. We observe as well that

$$\mathrm{sat}(n,\mathcal{R}_{\min}(H,K_2,\ldots,K_2))=\mathrm{sat}(n,H),$$

so that Problem 1 not only represents an interesting juxtaposition of classical Ramsey theory and graph saturation, but is also a direct extension of the problem of determining $\mathrm{sat}(n,H)$. Problem 1 is inspired by the following 1987 conjecture of Hanson and Toft [4].

Conjecture 1. *Let $r=r(K_{t_1},K_{t_2},\ldots,K_{t_k})$ be the standard Ramsey number for complete graphs. Then*

$$sat(n,\mathcal{R}_{\min}(K_{t_1},\ldots,K_{t_k}))=\begin{cases}\binom{n}{2} & n<r\\ \binom{r-2}{2}+(r-2)(n-r+2) & n\geq r.\end{cases}$$

In [1] it was shown that

$$\text{sat}(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$$

for $n \geq 54$, thereby verifying the first nontrivial case of Conjecture 1. At this time, however, it seems that a complete resolution of the Hanson-Toft conjecture remains elusive. As such, one goal of the study of Problem 1 is to develop a collection of techniques that might be useful in attacking Conjecture 1.

Here, we solve Problem 1 completely in the case where each H_i is a matching, and further completely characterize all saturated graphs of minimum size. Specifically, we prove the following.

Theorem 2. *If $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then*

$$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k).$$

If $m_i \geq 3$ for some i, then the unique saturated graphs of minimum size consist solely of vertex-disjoint triangles and independent vertices. If $m_i \leq 2$ for every i, then the graphs achieving equality are unions of edge-disjoint triangles and independent vertices.

As noted in [5], a result of Mader [6], which we utilize below, implies that the unique minimum saturated graph of order $n \geq 3m - 3$ for $H = mK_2$ is $(m-1)K_3 \cup (n - 3m + 3)K_1$. Hence, the minimum saturated graphs in Theorem 2 are precisely a union of m_iK_2-saturated graphs of minimum size. This provides an interesting contrast to both Conjecture 1 and the main result in [1] which posit and demonstrate, respectively, a stronger relationship between $r(K_{t_1}, K_{t_2}, \ldots, K_{t_k})$ and $sat(n, \mathcal{R}_{\min}(K_{t_1}, \ldots, K_{t_k}))$.

The proof of Theorem 2 uses *iterated recoloring*, a new technique that utilizes the structure of H_i-saturated graphs to gain insight into the properties of $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graphs. We describe this approach next.

1.1 Iterated Recoloring

Given graphs $G, H_1, \ldots, H_{k-1}$ and H_k, a k-edge coloring ϕ of G is an *$(H_1, \ldots, H_k)$-coloring* if G_ϕ contains no monochromatic copy of H_i in color i, but for any e in $\overline{G}$ and any $i \in [k]$, the addition of e to G in color i creates a monochromatic copy of H_i in color i. Central to our approach here is the following observation.

Observation 1. *If G is an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph, then every k-edge-coloring of G that contains no monochromatic copy of H_i in color i for any i is an $(H_1, \ldots, H_k)$-coloring. In particular, G has at least one $(H_1, \ldots, H_k)$-coloring.*

An $(H_1, \ldots, H_k)$-coloring of a graph G is *i-heavy* if for any edge e in G with color not equal to i, recoloring e with color i creates a monochromatic copy of H_i in color i. The next proposition connects the structure of H_i-saturated graphs with the monochromatic subgraph $G[i]$ in an i-heavy $(H_1, \ldots, H_k)$-coloring of G.

Lemma 3. *If G is an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph and ϕ is an i-heavy $(H_1, \ldots, H_k)$-coloring of G for some $i \in [k]$, then $G_\phi[i]$ is H_i-saturated.*

Proof. Throughout the proof, it suffices to treat $G[i]$ as an uncolored graph. As ϕ is an $(H_1, \ldots, H_k)$-coloring of G, it follows that $G[i]$ contains no subgraph isomorphic to H_i. It remains to prove that for any edge $e \in E(G[i])$, $G[i] + e$ has a subgraph isomorphic to H_i.

If $e \in E(G) - E(G[i])$, then $\phi(e) \neq i$. Because ϕ is i-heavy, changing e to color i in G_ϕ creates a copy of H_i in color i. Therefore, adding e to $G[i]$ creates a subgraph isomorphic to H_i. On the other hand, if $e \in E\left(\overline{G}\right)$, then the fact that ϕ is an $(H_1, \ldots, H_k)$-coloring of G implies that adding e to G_ϕ in color i creates a copy of H_i in color i. Consequently, $H_i \subseteq G[i] + e$. □

The general technique is as follows. Starting with an $(H_1, \ldots, H_k)$-coloring ϕ of an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph G, we iteratively recolor edges in G_ϕ to obtain a 1-heavy $(H_1, \ldots, H_k)$-coloring ϕ_1, and then recolor edges in G_{ϕ_1} to obtain a 2-heavy coloring ϕ_2, and so on until we have successively created i-heavy $(H_1, \ldots, H_k)$-colorings ϕ_i for every $i \in [k]$.

By Lemma 3, the monochromatic subgraph $G[i]$ corresponding to each ϕ_i is H_i-saturated. The goal is to then use any knowledge we may have about (uncolored) H_i-saturated graphs to force additional extra structure within G.

For instance, here we will use the following characterization of large enough mK_2-saturated graphs due to Mader [6]. A *dominating vertex* in a graph G of order n is a vertex of degree $n - 1$.

Theorem 4. *If G is an mK_2-saturated graph of order $n \geq 2m - 1$, then:*

1. *G is disconnected and every component is an odd clique, or*

2. *G has a dominating vertex v and $G - v$ is $(m-1)K_2$-saturated.*

2 Proof of Theorem 2

If $k = 1$, the result follows from the traditional saturation number for matchings, given in [5], so we may assume $k \geq 2$. Further, as $\text{sat}(\mathcal{R}_{min}(K_2, H_1, \ldots, H_k)) = \text{sat}(\mathcal{R}_{min}(H_1, \ldots, H_k))$, we may also assume that each $m_i \geq 2$. We begin by proving the upper bound in Theorem 2.

Proposition 5. *$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \ldots + m_k - k)$ whenever $n > 3(m_1 + \ldots + m_k - k)$.*

Proof. Let G be the vertex-disjoint union of $(m_1 + \ldots + m_k - k)$ triangles and $n - 3(m_1 + \ldots + m_k - k)$ independent vertices. We can create an $(m_1K_2, \ldots, m_kK_2)$-coloring ϕ of G by coloring the edges of $m_i - 1$ triangles with color i, for each i. A monochromatic matching can use at most one edge from each triangle, so for any i, the size of the largest matching in color i is $m_i - 1$.

Note that in any coloring of G containing no monochromatic m_iK_2 in color i for any i, each triangle is monochromatic and each color i is used in exactly $m_i - 1$ triangles. Further, there are at most $m_i - 1$ triangles containing an edge of color i, lest there exist an i-colored m_iK_2. Therefore, by the pigeonhole principle, the only way, up to isomorphism, to color G without creating one of the forbidden subgraphs is ϕ.

Consequently, for any $e = uv$ in $\overline{G}$, G_ϕ contains a copy of $(m_i - 1)K_2$ in color i that is disjoint from u and v. Given a k-edge coloring of $G + e$ in which G does not contain a copy of m_iK_2 in color i, it then follows that e lies in a monochromatic copy of $m_{\phi(e)}K_2$. Thus, G is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated. □

We note that if each $m_i = 2$, then there are minimum saturated graphs aside from kK_3. Indeed, let $n \geq 8$ and let G be the disjoint union of K_7 and $n-7$ isolated vertices. Note K_7 is the edge-disjoint union of seven triangles, so that any $(m_1K_2, \ldots, m_7K_2)$-coloring necessarily assigns a distinct color to each triangle. Then for any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$, so G is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated.

To prove the lower bound in Theorem 2, we will utilize the iterated recoloring technique described in Section 1.1. Assume that G is an $\mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)$-saturated graph of order $n > 3(m_1+\cdots+m_k-k)$ with at most $3(m_1+\cdots+m_k-k)$ edges. If G has a dominating vertex, then necessarily G is a star of order $3(m_1+\cdots+m_k-k)+1$, which is clearly not $\mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)$-saturated when $k \geq 2$. Hence we may assume that G contains no dominating vertex.

The following claims establish several important properties of G. The first follows immediately from Lemma 3 and the fact that G has no dominating vertex.

Proposition 6. *If ϕ is an i-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring of G, then $G[i]$ is the disjoint union of odd cliques.*

Next we show that no component of any $G[i]$ arising from an $(m_1K_2, \ldots, m_kK_2)$-coloring can have a cut edge.

Proposition 7. *If ϕ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of G, then each component of $G[i]$ is 2-edge-connected. In particular, each component C of $G[i]$ has at least $|V(C)|$ edges.*

Proof. Suppose ϕ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of G and that C is a component of $G[i]$ with cut-edge uv. As G_ϕ contains no m_i-matching in color i, every (m_i-1)-matching assigned color i in G_ϕ necessarily uses either u or v. Let $C - uv = C_1 \cup C_2$ for disjoint subgraphs C_1 and C_2 of C with $u \in C_1$ and $v \in C_2$.

Because G has no dominating vertex, there exist (not necessarily distinct) vertices x and y such that $ux, vy \in E(\overline{G})$. By the saturation of G, if we extend ϕ to $G + ux$ or $G + vy$ by assigning $\phi(ux) = i$ or $\phi(vy) = i$, respectively, then we create an m_i-matching in color i. Let M_u be an m_i-matching in color i in $G + ux$ that uses n_1 edges from $C_1 - u$ and n_2 edges from C_2. Then M_u restricted to G gives an (m_i-1)-matching that does not use u, and so uses v. Indeed, any matching on C_2 that has n_2 edges must use v.

Now let M_v be an m_i-matching in color i in $G + vy$. M_v restricted to G does not use v, so $C_2 - v$ contributes at most $n_2 - 1$ edges to M_v. Then C_1 contributes at least $n_1 + 1$ edges. Now, if we take the matching formed by restricting M_v to C_1 and M_u to $G - (C_2 \cup \{x\})$, then G has a matching in color i with at least $M_v[V(G) - (V(C_2) \cup \{x\})] + n_2 = m_i$ edges, a contradiction.

The assertion that C has at least as many edges as vertices then follows from the fact that C has no leaves.

□

Let ϕ be an $(H_1, \ldots, H_k)$-coloring of a graph G. An edge e in G is *inflexible* if changing the color of e to any $j \neq \phi(e)$ creates a monochromatic copy of H_j. The next proposition follows immediately from Proposition 7.

Proposition 8. *If ϕ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of G, and H is a component of some $G[i]$ that is isomorphic to a triangle, then every edge of H is inflexible.*

Let ϕ be an $(m_1K_2,\dots,m_kK_2)$-coloring of G, and let C be a component of $G_\phi[i]$. If ψ is a coloring of G obtained from ϕ by iteratively recoloring edges of G in a manner such that each successive coloring is an $(m_1K_2,\dots,m_kK_2)$-coloring, then we say that ψ is obtained from ϕ by *flexing*, or that we *flex* ϕ to ψ. In particular, it is always possible to flex to an i-heavy $(m_1K_2,\dots,m_kK_2)$-coloring of G from any other $(m_1K_2,\dots,m_kK_2)$-coloring of G.

Proposition 9. *Let ϕ be an $(m_1K_2,\dots,m_kK_2)$-coloring of G, and let C be a component of $G_\phi[i]$. If ψ is obtained from from ϕ by flexing, then $V(C)$ induces a component of $G_\psi[i]$.*

Proof. Suppose that there is some edge e such that recoloring e causes the order of C to increase or decrease in $G[i]$. If recoloring e to color i causes the order of C to increase, then e is necessarily a cut-edge in $G[i]$. On the other hand, if recoloring e causes the order of C to decrease, then prior to recoloring, e was a cut-edge in $G[i]$. In either case, we have contradicted Proposition 7, and the proposition follows by induction. □

Let ϕ be an $(m_1K_2,\dots,m_kK_2)$-coloring of G and flex ϕ to a 1-heavy $(m_1K_2,\dots,m_kK_2)$-coloring ϕ_1. For $2 \le i \le k$, flex ϕ_{i-1} to an i-heavy $(m_1K_2,\dots,m_kK_2)$-coloring ϕ_i. Consider then the nontrivial components of $G_{\phi_i}[i]$, all of which are odd cliques by Proposition 6. In particular, suppose that these components have order $2x_j+1$ for $1 \le j \le \ell$. Then, as ϕ_i is an $(m_1K_2,\dots,m_kK_2)$-coloring, we have that $x_1+\cdots+x_\ell = m_i - 1$. Further, since the components of G_i do not change order via flexing, a component C of order $2x+1$ in $G_{\phi_j}[i]$ must have a maximum matching of size x.

Propositions 6 and 9 imply that a set X of vertices in G induces a component of $G_{\phi_i}[i]$ if and only if X induces a component of $G_{\phi_j}[i]$ for all $i,j \in [k]$. This, in turn, implies that if ϕ' and ϕ'' are i-heavy colorings obtained via flexing from ϕ, then $G_{\phi'}[i] = G_{\phi''}[i]$. This yields the following proposition.

Proposition 10. *Let C be a component of $G_{\phi_i}[i]$. Then there are at least $|V(C)|$ edges e in C such that $\phi_j(e) = i$ for all $1 \le j \le k$.*

Proof. Let $S \subset E(C)$ be those edges e in C such that $\{\phi_j(e) \,:\, 1 \le j \le k\} = \{i\}$ and suppose that $|S| < |V(C)|$. Every edge of C that is not in S lies in some component C' of $G_{\phi_j}[j]$ for some $j \ne i$. Iteratively recoloring each $e \notin S$ with any such j does not create a matching of size m_ℓ in color ℓ for any ℓ, as all edges colored ℓ lie within some component of $G_{\phi_\ell}[\ell]$. However, this means that at most $|S| < |V(C)|$ edges of C remain colored with color i, contradicting Proposition 9. □

Our final proposition shows that no edge in G receives more than two colors under $\phi_1,\dots,\phi_k$.

Proposition 11. *If Q is a component of $G_{\phi_i}[i]$ on $2m+1$ vertices, with $m \ge 1$, then any edge of Q is assigned at most 2 colors under $\phi_1,\dots,\phi_k$. Furthermore, if Q is a triangle, then every edge of Q is inflexible in every G_{ϕ_i}.*

Proof. Note first that if $m = 1$, so that Q is a triangle, then this is the result of Proposition 8. Hence we will assume that $m \ge 2$.

Suppose Q is a component of $G_{\phi_1}[1]$, and an edge $uv \in E(Q)$ appears in components Q_2 and Q_3 of $G_{\phi_2}[2]$ and $G_{\phi_3}[3]$, respectively. Recall that by Proposition 6, Q_2 and Q_3 are necessarily odd cliques.

Let $V(Q) - \{u,v\} = \{x_1, x_2, \dots, x_{2m-1}\}$. First, we define a coloring ψ' of Q.

$$\psi'(e) := \begin{cases} 2 & \text{if } e = x_2x_j \\ 3 & \text{if } e = x_3x_j \text{ with } j \neq 2 \\ 1 & \text{otherwise} \end{cases}$$

Now:

$$\psi(e) := \begin{cases} \phi(e) & \text{if } e \notin Q \cup Q_2 \cup Q_3 \\ 1 & \text{if } e \text{ is in } Q \cup Q_2 \cup Q_3 \text{ and incident to } u \text{ or } v. \\ \psi'(e) & \text{if } e \text{ is not incident to } u, v \text{ and } e \text{ is in } Q \\ 2 & e \text{ is not incident to } u \text{ or } v\text{, and } e \in Q_2 \setminus Q_3 \\ 3 & e \text{ is not incident to } u \text{ or } v\text{, and } e \in Q_3 \end{cases}$$

In this coloring, the $(2m-3)$ vertices $\{x_1, x_4, \ldots, x_{2m-1}\}$ form a clique of color 1, contributing at most $m-2$ edges to any matching in color 1. Further, edges incident to u or v also contribute at most two matching edges, so any matching in color 1 has at most m edges with an endpoint in Q. As Proposition 9 implies that the other ℓ nontrivial components of $G_\psi[1] - V(Q)$ are odd cliques with total order $2m_1 - 2m + \ell - 2$, the maximum size of a matching with color 1 in G_ψ is $m_1 - 1$.

Let Q_2 have $2n_2+1$ vertices, and let Q_3 have $2n_3+1$ vertices. Note that in G_{ϕ_1}, Q_2 contributes n_2 edges to any maximum monochromatic matching of color 2 and Q_3 contributes n_3 edges to any maximum monochromatic matching of color 3. As we have recolored all edges in $Q \cup Q_2 \cup Q_3$ that are incident to u or v with color 1, for color $i \in \{2, 3\}$, $Q_i - u - v$ contains a matching of size $n_i - 1$. One more edge of color i incident with x_i completes a matching of size at most n_i in $Q \cup Q_2 \cup Q_3$. Outside $Q \cup Q_1 \cup Q_2$, $\psi = \phi$, so ψ is a $(H_1, \ldots, H_k)$-coloring.

If x is a vertex in G that is not adjacent to u, then adding the edge ux to G in color 1 does not increase the size of a maximum 1-colored matching. Thus G is not $\mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)$-saturated, a contradiction. □

We are now ready to prove Theorem 2.

Proof. Let G and $\phi_1, \ldots, \phi_k$ be as given above, and further assume that

$$|E(G)| = \mathrm{sat}(n, \mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \cdots + m_k - k).$$

For each i, we let $Q_{i,1}, \ldots, Q_{i,p_i}$ be the (clique) components of $G_{\phi_i}[i]$, and suppose that each $Q_{i,j}$ has $2t_{i,j}+1$ vertices. Recall that $\sum_{j=1}^{p_i} t_{i,j} = m_i - 1$.

For any $e \in E(G)$, we define $w(e) = |\{\phi_i(e) \ : \ 1 \leq i \leq k\}|$. That is, $w(e)$ is the number of colors assigned to e by the heavy colorings $\phi_1, \ldots, \phi_k$. Note

$$|E(G)| = \sum_{i=1}^{k} \sum_{e \in G_{\phi_i}[i]} \frac{1}{w(e)}.$$

By Proposition 11, $w(e) \leq 2$ for every edge of G. Further, by Proposition 10, $w(e) = 1$ for at least $|V(Q)|$ edges of Q. Therefore,

$$
\begin{aligned}
|E(G)| &= \sum_{i=1}^{k} \sum_{e \in G[i]} \frac{1}{w(e)} \\
&\geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} \left((2t_{i,j}+1) + \frac{1}{2} \left[\binom{2t_{i,j}+1}{2} - (2t_{i,j}+1) \right] \right) \qquad (1) \\
&\geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} 3t_{i,j} = \sum_{i=1}^{k} 3(m_i - 1) = 3(m_1 + \ldots + m_k - k).
\end{aligned}
$$

We therefore conclude that

$$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k) - k.$$

Additionally, equality holds in all equations above, leading us to conclude that every component of every $G_{\phi_i}[i]$ is a triangle. By Proposition 8, also every component of every $G_{\phi_i}[j]$ is a triangle.

It remains only to show that if $m_i \geq 3$ for at least one i, then G consists of triangles that are vertex disjoint. Suppose not. Then there exists at least one "bow-tie" B: a subgraph of G consisting of two triangles that share one vertex. We can create an $(H_1, \ldots, H_k)$-coloring ϕ of G by assigning color i to $m_i - 1$ of the edge-disjoint triangles in a triangle decomposition of G. Let ϕ be a such a coloring, in which both triangles of B are assigned color i. If we flex ϕ to be i-heavy, then Proposition 6 implies that the vertices of B must lie in a clique on at least five vertices. However, as equality holds throughout (1) and ϕ was selected arbitrarily, each component of $G[i]$ under any valid coloring is a triangle, a contradiction. □

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