Ramsey-Minimal Saturation Numbers for Matchings
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Abstract

Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-saturated if no element of $\mathcal{F}$ is a subgraph of $G$, but for any edge $e$ in $G$, some element of $\mathcal{F}$ is a subgraph of $G + e$. Let $\text{sat}(n, \mathcal{F})$ denote the minimum number of edges in an $\mathcal{F}$-saturated graph of order $n$, which we refer to as the saturation number or saturation function of $\mathcal{F}$. If $\mathcal{F} = \{F\}$, then we instead say that $G$ is $F$-saturated and write $\text{sat}(n,F)$.

For graphs $G,H_1,\ldots,H_k$, we write that $G \rightarrow (H_1,\ldots,H_k)$ if every $k$-coloring of $E(G)$ contains a monochromatic copy of $H_i$ in color $i$ for some $i$. A graph $G$ is $(H_1,\ldots,H_k)$-Ramsey-minimal if $G \rightarrow (H_1,\ldots,H_k)$ but for any $e \in G$, $(G - e) \not\rightarrow (H_1,\ldots,H_k)$. Let $\mathcal{R}_{\text{min}}(H_1,\ldots,H_k)$ denote the family of $(H_1,\ldots,H_k)$-Ramsey-minimal graphs.

In this paper, motivated in part by a conjecture of Hanson and Toft [Edge-colored saturated graphs, J. Graph Theory 11 (1987), 191–196], we prove that
\[
\text{sat}(n, \mathcal{R}_{\text{min}}(m_1K_2,\ldots,m_kK_2)) = 3(m_1 + \ldots + m_k - k)
\]
for $m_1,\ldots,m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, and we also characterize the saturated graphs of minimum size. The proof of this result uses a new technique, iterated recoloring, which takes advantage of the structure of $H_i$-saturated graphs to determine the saturation number of $\mathcal{R}_{\text{min}}(H_1,\ldots,H_k)$.

Keywords: saturated graph, Ramsey-minimal graph, matching

1 Introduction

All graphs considered in this paper are simple, undirected and finite. For any undefined terminology or notation, please see [7]. Given an edge coloring $\phi$ of a graph $G$ let $G_\phi$ denote the edge-colored graph obtained by applying $\phi$ to $G$, and let $G_\phi[i]$ denote the spanning subgraph of $G_\phi$ induced by all edges of color $i$. When the context is clear, we will simply write $G$ and $G[i]$ in place of the more cumbersome $G_\phi$ and $G_\phi[i]$.

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Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-saturated if no element of $\mathcal{F}$ is a subgraph of $G$, but for any edge $e$ in $G$, some element of $\mathcal{F}$ is a subgraph of $G + e$. If $\mathcal{F} = \{F\}$, then we say that $G$ is $F$-saturated. The classical extremal function $\text{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-saturated graph of order $n$.

In this paper, we are concerned with $\text{sat}(n, \mathcal{F})$, the minimum number of edges in an $\mathcal{F}$-saturated graph of order $n$. We refer to $\text{sat}(n, \mathcal{F})$ as the saturation number or saturation function of $\mathcal{F}$. This parameter was introduced by Erdős, Hajnal and Moon in [2], wherein they determined $\text{sat}(n, K_t)$ and characterized the unique saturated graphs of minimum size. Here “$\lor$” denotes the standard graph join.

**Theorem 1.** If $n$ and $t$ are positive integers such that $n \geq t$, then

$$\text{sat}(n, K_t) = \left(\frac{t - 2}{2}\right) + (t - 2)(n - t - 2).$$

Furthermore, $K_{t-2} \lor \overline{K}_{n-t+2}$ is the unique $K_t$-saturated graph of order $n$ with minimum size.

Subsequently, $\text{sat}(n, \mathcal{F})$ has been determined for a number of families of graphs and hypergraphs. We refer the interested reader to the dynamic survey of Faudree, Faudree and Schmitt [3], which gives a thorough overview of the area.

For graphs $G, H_1, \ldots, H_k$, we write that $G \rightarrow (H_1, \ldots, H_k)$ if every $k$-coloring of $E(G)$ contains a monochromatic copy of $H_i$ in color $i$ for some $i$. The (classical) Ramsey number $r(H_1, \ldots, H_K)$ is the smallest positive integer $n$ such that $K_n \rightarrow (H_1, \ldots, H_k)$. A graph $G$ is $(H_1, \ldots, H_k)$-Ramsey-minimal if $G \rightarrow (H_1, \ldots, H_k)$ but for any $e \in G$, $(G - e) \nrightarrow (H_1, \ldots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_k)$-Ramsey-minimal graphs.

Here we are interested in the following general problem.

**Problem 1.** Let $H_1, \ldots, H_k$ be graphs, each with at least one edge. Determine

$$\text{sat}(n, \mathcal{R}_{\min}(H_1, \ldots, H_k)).$$

It is straightforward to prove that $G \rightarrow (H_1, \ldots, H_k)$ if and only if $G$ contains an $(H_1, \ldots, H_k)$-Ramsey-minimal subgraph. Hence Problem 1 is equivalent to finding the minimum size of a graph $G$ of order $n$ such that there is some $k$-edge-coloring of $G$ that contains no copy of $H_i$ in color $i$ for any $i$, yet for any $e \in G$ every $k$-edge-coloring of $G + e$ contains a monochromatic copy of $H_i$ in color $i$ for some $i$. We observe as well that

$$\text{sat}(n, \mathcal{R}_{\min}(H, K_2, \ldots, K_2)) = \text{sat}(n, H),$$

so that Problem 1 not only represents an interesting juxtaposition of classical Ramsey theory and graph saturation, but is also a direct extension of the problem of determining $\text{sat}(n, H)$. Problem 1 is inspired by the following 1987 conjecture of Hanson and Toft [4].

**Conjecture 1.** Let $r = r(K_{t_1}, K_{t_2}, \ldots, K_{t_k})$ be the standard Ramsey number for complete graphs. Then

$$\text{sat}(n, \mathcal{R}_{\min}(K_{t_1}, \ldots, K_{t_k})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r - 2)(n - r + 2) & n \geq r. \end{cases}$$
In [1] it was shown that 
\[ \text{sat}(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10 \]
for \( n \geq 54 \), thereby verifying the first nontrivial case of Conjecture 1. At this time, however, it seems that a complete resolution of the Hanson-Toft conjecture remains elusive. As such, one goal of the study of Problem 1 is to develop a collection of techniques that might be useful in attacking Conjecture 1.

Here, we solve Problem 1 completely in the case where each \( H_i \) is a matching, and further completely characterize all saturated graphs of minimum size. Specifically, we prove the following.

**Theorem 2.** If \( m_1, \ldots, m_k \geq 1 \) and \( n > 3(m_1 + \ldots + m_k - k) \), then 
\[ \text{sat}(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k). \]

If \( m_i \geq 3 \) for some \( i \), then the unique saturated graphs of minimum size consist solely of vertex-disjoint triangles and independent vertices. If \( m_i \leq 2 \) for every \( i \), then the graphs achieving equality are unions of edge-disjoint triangles and independent vertices.

As noted in [5], a result of Mader [6], which we utilize below, implies that the unique minimum saturated graph of order \( n \geq 3m - 3 \) for \( H = mK_2 \) is \((m - 1)K_3 \cup (n - 3m + 3)K_1\). Hence, the minimum saturated graphs in Theorem 2 are precisely a union of \( m_iK_2 \)-saturated graphs of minimum size. This provides an interesting contrast to both Conjecture 1 and the main result in [1] which posit and demonstrate, respectively, a stronger relationship between \( r(K_{t_1}, K_{t_2}, \ldots, K_{t_k}) \) and \( \text{sat}(n, \mathcal{R}_{\min}(K_{t_1}, \ldots, K_{t_k})) \).

The proof of Theorem 2 uses **iterated recoloring**, a new technique that utilizes the structure of \( H_i \)-saturated graphs to gain insight into the properties of \( \mathcal{R}_{\min}(H_1, \ldots, H_k) \)-saturated graphs. We describe this approach next.

### 1.1 Iterated Recoloring

Given graphs \( G, H_1, \ldots, H_{k-1} \) and \( H_k \), a \( k \)-edge coloring \( \phi \) of \( G \) is an \((H_1, \ldots, H_k)\)-coloring if \( G_\phi \) contains no monochromatic copy of \( H_i \) in color \( i \), but for any \( e \in G \) and any \( i \in [k] \), the addition of \( e \) to \( G \) in color \( i \) creates a monochromatic copy of \( H_i \) in color \( i \). Central to our approach here is the following observation.

**Observation 1.** If \( G \) is an \( \mathcal{R}_{\min}(H_1, \ldots, H_k) \)-saturated graph, then every \( k \)-edge-coloring of \( G \) that contains no monochromatic copy of \( H_i \) in color \( i \) for any \( i \) is an \((H_1, \ldots, H_k)\)-coloring. In particular, \( G \) has at least one \((H_1, \ldots, H_k)\)-coloring.

An \((H_1, \ldots, H_k)\)-coloring of a graph \( G \) is \( i \)-heavy if for any edge \( e \) in \( G \) with color not equal to \( i \), recoloring \( e \) with color \( i \) creates a monochromatic copy of \( H_i \) in color \( i \). The next proposition connects the structure of \( H_i \)-saturated graphs with the monochromatic subgraph \( G[i] \) in an \( i \)-heavy \((H_1, \ldots, H_k)\)-coloring of \( G \).

**Lemma 3.** If \( G \) is an \( \mathcal{R}_{\min}(H_1, \ldots, H_k) \)-saturated graph and \( \phi \) is an \( i \)-heavy \((H_1, \ldots, H_k)\)-coloring of \( G \) for some \( i \in [k] \), then \( G_\phi[i] \) is \( H_i \)-saturated.
Proof. Throughout the proof, it suffices to treat \( G[i] \) as an uncolored graph. As \( \phi \) is an \((H_1, \ldots, H_k)\)-coloring of \( G \), it follows that \( G[i] \) contains no subgraph isomorphic to \( H_i \). It remains to prove that for any edge \( e \in E(G[i]) \), \( G[i] + e \) has a subgraph isomorphic to \( H_i \).

If \( e \in E(G) - E(G[i]) \), then \( \phi(e) \neq i \). Because \( \phi \) is \( i \)-heavy, changing \( e \) to color \( i \) in \( G_\phi \) creates a copy of \( H_i \) in color \( i \). Therefore, adding \( e \) to \( G[i] \) creates a subgraph isomorphic to \( H_i \). On the other hand, if \( e \in E(G) \), then the fact that \( \phi \) is an \((H_1, \ldots, H_k)\)-coloring of \( G \) implies that adding \( e \) to \( G_\phi \) in color \( i \) creates a copy of \( H_i \) in color \( i \). Consequently, \( H_i \subseteq G[i] + e \).

The general technique is as follows. Starting with an \((H_1, \ldots, H_k)\)-coloring \( \phi \) of an \( \mathcal{R}_{\min}(H_1, \ldots, H_k) \)-saturated graph \( G \), we iteratively recolor edges in \( G_\phi \) to obtain a 1-heavy \((H_1, \ldots, H_k)\)-coloring \( \phi_1 \), and then recolor edges in \( G_{\phi_1} \) to obtain a 2-heavy \((H_1, \ldots, H_k)\)-coloring \( \phi_2 \), and so on until we have successively created \( i \)-heavy \((H_1, \ldots, H_k)\)-colorings \( \phi_i \) for every \( i \in [k] \).

By Lemma 3, the monochromatic subgraph \( G[i] \) corresponding to each \( \phi_i \) is \( H_i \)-saturated. The goal is to then use any knowledge we may have about (uncolored) \( H_i \)-saturated graphs to force additional extra structure within \( G \).

For instance, here we will use the following characterization of large enough \( mK_2 \)-saturated graphs due to Mader [6]. A dominating vertex in a graph \( G \) of order \( n \) is a vertex of degree \( n - 1 \).

**Theorem 4.** If \( G \) is an \( mK_2 \)-saturated graph of order \( n \geq 2m - 1 \), then:

1. \( G \) is disconnected and every component is an odd clique, or
2. \( G \) has a dominating vertex \( v \) and \( G - v \) is \((m - 1)K_2\)-saturated.

## 2 Proof of Theorem 2

If \( k = 1 \), the result follows from the traditional saturation number for matchings, given in [5], so we may assume \( k \geq 2 \). Further, as \( \text{sat}(\mathcal{R}_{\min}(K_2, H_1, \ldots, H_k)) = \text{sat}(\mathcal{R}_{\min}(H_1, \ldots, H_k)) \), we may also assume that each \( m_i \geq 2 \). We begin by proving the upper bound in Theorem 2.

**Proposition 5.** \( \text{sat}(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \ldots + m_k - k) \) whenever \( n > 3(m_1 + \ldots + m_k - k) \).

**Proof.** Let \( G \) be the vertex-disjoint union of \((m_1 + \ldots + m_k - k)\) triangles and \( n - 3(m_1 + \ldots + m_k - k) \) independent vertices. We can create an \((m_1K_2, \ldots, m_kK_2)\)-coloring \( \phi \) of \( G \) by coloring the edges of \( m_i - 1 \) triangles with color \( i \), for each \( i \). A monochromatic matching can use at most one edge from each triangle, so for any \( i \), the size of the largest matching in color \( i \) is \( m_i - 1 \).

Note that in any coloring of \( G \) containing no monochromatic \( m_iK_2 \) in color \( i \) for any \( i \), each triangle is monochromatic and each color \( i \) is used in exactly \( m_i - 1 \) triangles. Further, there are at most \( m_i - 1 \) triangles containing an edge of color \( i \), lest there exist an \( i \)-colored \( m_iK_2 \). Therefore, by the pigeonhole principle, the only way, up to isomorphism, to color \( G \) without creating one of the forbidden subgraphs is \( \phi \).

Consequently, for any \( e = uv \) in \( G \), \( G_\phi \) contains a copy of \((m_i - 1)K_2 \) in color \( i \) that is disjoint from \( u \) and \( v \). Given a \( k \)-edge coloring of \( G + e \) in which \( G \) does not contain a copy of \( m_iK_2 \) in color \( i \), it then follows that \( e \) lies in a monochromatic copy of \( m_{\phi(e)}K_2 \). Thus, \( G \) is \( \mathcal{R}_{\min}(H_1, \ldots, H_k) \)-saturated. \( \square \)
We note that if each $m_i = 2$, then there are minimum saturated graphs aside from $kK_3$. Indeed, let $n \geq 8$ and let $G$ be the disjoint union of $K_7$ and $n - 7$ isolated vertices. Note $K_7$ is the edge-disjoint union of seven triangles, so that any $(m_1K_2, \ldots, m_7K_2)$-coloring necessarily assigns a distinct color to each triangle. Then for any $e \in E(G)$, $G + e \rightarrow (H_1, \ldots, H_k)$, so $G$ is $R_{\min}(H_1, \ldots, H_k)$-saturated.

To prove the lower bound in Theorem 2, we will utilize the iterated recoloring technique described in Section 1.1. Assume that $G$ is an $R_{\min}(m_1K_2, \ldots, m_kK_2)$-saturated graph of order $n > 3(m_1 + \cdots + m_k - k)$ with at most $3(m_1 + \cdots + m_k - k)$ edges. If $G$ has a dominating vertex, then necessarily $G$ is a star of order $3(m_1 + \cdots + m_k - k) + 1$, which is clearly not $R_{\min}(m_1K_2, \ldots, m_kK_2)$-saturated when $k \geq 2$. Hence we may assume that $G$ contains no dominating vertex.

The following claims establish several important properties of $G$. The first follows immediately from Lemma 3 and the fact that $G$ has no dominating vertex.

**Proposition 6.** If $\phi$ is an $i$-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$, then $G[i]$ is the disjoint union of odd cliques.

Next we show that no component of any $G[i]$ arising from an $(m_1K_2, \ldots, m_kK_2)$-coloring can have a cut edge.

**Proposition 7.** If $\phi$ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$, then each component of $G[i]$ is 2-edge-connected. In particular, each component $C$ of $G[i]$ has at least $|V(C)|$ edges.

**Proof.** Suppose $\phi$ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$ and that $C$ is a component of $G[i]$ with cut-edge $uv$. As $G_\phi$ contains no $m_i$-matching in color $i$, every $(m_i - 1)$-matching assigned color $i$ in $G_\phi$ necessarily uses either $u$ or $v$. Let $C = uv = C_1 \cup C_2$ for disjoint subgraphs $C_1$ and $C_2$ of $C$ with $u \in C_1$ and $v \in C_2$.

Because $G$ has no dominating vertex, there exist (not necessarily distinct) vertices $x$ and $y$ such that $ux, vy \in E(G)$. By the saturation of $G$, if we extend $\phi$ to $G + ux$ or $G + vy$ by assigning $\phi(ux) = i$ or $\phi(vy) = i$, respectively, then we create an $m_i$-matching in color $i$. Let $M_u$ be an $m_i$-matching in color $i$ in $G + ux$ that uses $n_1$ edges from $C_1 - u$ and $n_2$ edges from $C_2$. Then $M_u$ restricted to $G$ gives an $(m_i - 1)$-matching that does not use $u$, and so uses $v$. Indeed, any matching on $C_2$ that has $n_2$ edges must use $v$.

Now let $M_v$ be an $m_i$-matching in color $i$ in $G + vy$. $M_v$ restricted to $G$ does not use $v$, so $C_2 - v$ contributes at most $n_2 - 1$ edges to $M_v$. Then $C_1$ contributes at least $n_1 + 1$ edges. Now, if we take the matching formed by restricting $M_u$ to $C_1$ and $M_u$ to $G - (C_2 \cup \{x\})$, then $G$ has a matching in color $i$ with at least $M_v[|G| - (|V(C_2) \cup \{x\}|) + n_2 = m_i$ edges, a contradiction.

The assertion that $C$ has at least as many edges as vertices then follows from the fact that $C$ has no leaves.

Let $\phi$ be an $(H_1, \ldots, H_k)$-coloring of a graph $G$. An edge $e$ in $G$ is inflexible if changing the color of $e$ to any $j \neq \phi(e)$ creates a monochromatic copy of $H_j$. The next proposition follows immediately from Proposition 7.

**Proposition 8.** If $\phi$ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$, and $H$ is a component of some $G[i]$ that is isomorphic to a triangle, then every edge of $H$ is inflexible.
Let $\phi$ be an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$, and let $C$ be a component of $G_\phi[i]$. If $\psi$ is a coloring of $G$ obtained from $\phi$ by iteratively recoloring edges of $G$ in a manner such that each successive coloring is an $(m_1K_2, \ldots, m_kK_2)$-coloring, then we say that $\psi$ is obtained from $\phi$ by flexing, or that we flex $\phi$ to $\psi$. In particular, it is always possible to flex to an $i$-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$ from any other $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$.

**Proposition 9.** Let $\phi$ be an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$, and let $C$ be a component of $G_\phi[i]$. If $\psi$ is obtained from $\phi$ by flexing, then $V(C)$ induces a component of $G_\psi[i]$.

**Proof.** Suppose that there is some edge $e$ such that recoloring $e$ causes the order of $C$ to increase or decrease in $G[i]$. If recoloring $e$ to color $i$ causes the order of $C$ to increase, then $e$ is necessarily a cut-edge in $G[i]$. On the other hand, if recoloring $e$ causes the order of $C$ to decrease, then prior to recoloring, $e$ was a cut-edge in $G[i]$. In either case, we have contradicted Proposition 7, and the proposition follows by induction. □

Let $\phi$ be an $(m_1K_2, \ldots, m_kK_2)$-coloring of $G$ and flex $\phi$ to a 1-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring $\phi_1$. For $2 \leq i \leq k$, flex $\phi_{i-1}$ to an $i$-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring $\phi_i$. Consider then the nontrivial components of $G_{\phi_i}[i]$, all of which are odd cliques by Proposition 6. In particular, suppose that these components have order $2x + 1$ for $1 \leq j \leq \ell$. Then, as $\phi_1$ is an $(m_1K_2, \ldots, m_kK_2)$-coloring, we have that $x_1 + \cdots + x_\ell = m_i - 1$. Further, since the components of $G_i$ do not change order via flexing, a component $C$ of order $2x + 1$ in $G_{\phi_j}[i]$ must have a maximum matching of size $x$.

Propositions 6 and 9 imply that a set $X$ of vertices in $G$ induces a component of $G_{\phi_i}[i]$ if and only if $X$ induces a component of $G_{\phi_j}[i]$ for all $i, j \in [k]$. This, in turn, implies that if $\phi'$ and $\phi''$ are $i$-heavy colorings obtained via flexing from $\phi$, then $G_{\phi'}[i] = G_{\phi''}[i]$. This yields the following proposition.

**Proposition 10.** Let $C$ be a component of $G_{\phi_i}[i]$. Then there are at least $|V(C)|$ edges in $C$ such that $\phi_j(e) = i$ for all $1 \leq j \leq k$.

**Proof.** Let $S \subset E(C)$ be those edges $e$ in $C$ such that $\{\phi_j(e) : 1 \leq j \leq k\} = \{i\}$ and suppose that $|S| < |V(C)|$. Every edge of $C$ that is not in $S$ lies in some component $C'$ of $G_{\phi_j}[j]$ for some $j \neq i$. Iteratively recoloring each $e \notin S$ with any such $j$ does not create a matching of size $m_i$ in color $\ell$ for any $\ell$, as all edges colored $\ell$ lie within some component of $G_{\phi_j}[\ell]$. However, this means that at most $|S| < |V(C)|$ edges of $C$ remain colored with color $i$, contradicting Proposition 9. □

Our final proposition shows that no edge in $G$ receives more than two colors under $\phi_1, \ldots, \phi_k$.

**Proposition 11.** If $Q$ is a component of $G_{\phi_i}[i]$ on $2m + 1$ vertices, with $m \geq 1$, then any edge of $Q$ is assigned at most 2 colors under $\phi_1, \ldots, \phi_k$. Furthermore, if $Q$ is a triangle, then every edge of $Q$ is inflexible in every $G_{\phi_i}$.

**Proof.** Note first that if $m = 1$, so that $Q$ is a triangle, then this is the result of Proposition 8. Hence we will assume that $m \geq 2$.

Suppose $Q$ is a component of $G_{\phi_1}[1]$, and an edge $uv \in E(Q)$ appears in components $Q_2$ and $Q_3$ of $G_{\phi_2}[2]$ and $G_{\phi_3}[3]$, respectively. Recall that by Proposition 6, $Q_2$ and $Q_3$ are necessarily odd cliques.

Let $V(Q) - \{u, v\} = \{x_1, x_2, \ldots, x_{2m-1}\}$. First, we define a coloring $\psi'$ of $Q$.  

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\[ \psi'(e) := \begin{cases} 2 & \text{if } e = x_2x_j \\ 3 & \text{if } e = x_3x_j \text{ with } j \neq 2 \\ 1 & \text{otherwise} \end{cases} \]

Now:
\[ \phi(e) := \begin{cases} \psi'(e) & \text{if } e \notin Q \cup Q_2 \cup Q_3 \\ 1 & \text{if } e \text{ is in } Q \cup Q_2 \cup Q_3 \text{ and incident to } u \text{ or } v. \end{cases} \]

In this coloring, the \((2m - 3)\) vertices \(\{x_1, x_4, \ldots, x_{2m-1}\}\) form a clique of color 1, contributing at most \(m - 2\) edges to any matching in color 1. Further, edges incident to \(u\) or \(v\) also contribute at most two matching edges, so any matching in color 1 has at most \(m\) edges with an endpoint in \(Q\). As Proposition 9 implies that the other \(\ell\) nontrivial components of \(G_\emptyset[1] - V(Q)\) are odd cliques with total order \(2m_1 - 2m + \ell - 2\), the maximum size of a matching with color 1 in \(G_\emptyset\) is \(m_1 - 1\).

Let \(Q_2\) have \(2n_2 + 1\) vertices, and let \(Q_3\) have \(2n_3 + 1\) vertices. Note that in \(G_{\phi_1}\), \(Q_2\) contributes \(n_2\) edges to any maximum monochromatic matching of color 2 and \(Q_3\) contributes \(n_3\) edges to any maximum monochromatic matching of color 3. As we have recolored all edges in \(Q \cup Q_2 \cup Q_3\) that are incident to \(u\) or \(v\) with color 1, for color \(i \in \{2, 3\}\), \(Q_i - u - v\) contains a matching of size \(n_i - 1\). One more edge of color \(i\) incident with \(x_i\) completes a matching of size at most \(n_i\) in \(Q \cup Q_2 \cup Q_3\). Outside \(Q \cup Q_1 \cup Q_2\), \(\psi = \phi\), so \(\psi\) is a \((H_1, \ldots, H_k)\)-coloring.

If \(x\) is a vertex in \(G\) that is not adjacent to \(u\), then adding the edge \(ux\) to \(G\) in color 1 does not increase the size of a maximum 1-colored matching. Thus \(G\) is not \(\mathcal{R}_\min(m_1K_2, \ldots, m_kK_2)\)-saturated, a contradiction. \(\square\)

We are now ready to prove Theorem 2.

**Proof.** Let \(G\) and \(\phi_1, \ldots, \phi_k\) be as given above, and further assume that
\[ |E(G)| = \text{sat}(n, \mathcal{R}_\min(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \cdots + m_k - k). \]

For each \(i\), we let \(Q_{i,1}, \ldots, Q_{i,p_i}\) be the (clique) components of \(G_{\phi_i}[i]\), and suppose that each \(Q_{i,j}\) has \(2t_{i,j} + 1\) vertices. Recall that \(\sum_{j=1}^{p_i} t_{i,j} = m_i - 1\).

For any \(e \in E(G)\), we define \(w(e) = |\{\phi_i(e) : 1 \leq i \leq k\}|\). That is, \(w(e)\) is the number of colors assigned to \(e\) by the heavy colorings \(\phi_1, \ldots, \phi_k\). Note
\[ |E(G)| = \sum_{i=1}^{k} \sum_{e \in G_{\phi_i}[i]} \frac{1}{w(e)}. \]

By Proposition 11, \(w(e) \leq 2\) for every edge of \(G\). Further, by Proposition 10, \(w(e) = 1\) for at least \(|V(Q)|\) edges of \(Q\). Therefore,
\[ |E(G)| = \sum_{i=1}^{k} \sum_{e \in G[i]} \frac{1}{w(e)} \geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} \left( (2t_{i,j} + 1) + \frac{1}{2} \left( \frac{(2t_{i,j} + 1)}{2} - (2t_{i,j} + 1) \right) \right) \geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} 3t_{i,j} = \sum_{i=1}^{k} 3(m_i - 1) = 3(m_1 + \ldots + m_k - k). \] (1)

We therefore conclude that

\[ sat(n, \mathcal{R}_{\text{min}}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k) - k. \]

Additionally, equality holds in all equations above, leading us to conclude that every component of every \( G_{\phi_i}[j] \) is a triangle. By Proposition 8, also every component of every \( G_{\phi_i}[j] \) is a triangle.

It remains only to show that if \( m_i \geq 3 \) for at least one \( i \), then \( G \) consists of triangles that are vertex disjoint. Suppose not. Then there exists at least one "bow-tie" \( B \): a subgraph of \( G \) consisting of two triangles that share one vertex. We can create an \((H_1, \ldots, H_k)\)-coloring \( \phi \) of \( G \) by assigning color \( i \) to \( m_i - 1 \) of the edge-disjoint triangles in a triangle decomposition of \( G \). Let \( \phi \) be a such a coloring, in which both triangles of \( B \) are assigned color \( i \). If we flex \( \phi \) to be \( i \)-heavy, then Proposition 6 implies that the vertices of \( B \) must lie in a clique on at least five vertices. However, as equality holds throughout (1) and \( \phi \) was selected arbitrarily, each component of \( G[i] \) under any valid coloring is a triangle, a contradiction. \qed

**References**


