Average Connectivity and Average Edge-connectivity in Graphs

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Abstract

Connectivity and edge-connectivity of a graph measure the difficulty of breaking the graph apart, but they are very much affected by local aspects like vertex degree. Average connectivity (and analogously, average edge-connectivity) has been introduced to give a more refined measure of the global “amount” of connectivity. In this paper, we prove a relationship between the average connectivity and the matching number in all graphs. We also give the best lower bound for the average edge-connectivity over $n$-vertex connected cubic graphs, and we characterize the graphs where equality holds. In addition, we show that this family has the fewest perfect matchings among cubic graphs that have perfect matchings.

1 Introduction

A graph $G$ is $k$-connected if it has more than $k$ vertices and every subgraph obtained by deleting fewer than $k$ vertices is connected. The connectivity of $G$, written $\kappa(G)$, is the maximum $k$ such that $G$ is $k$-connected. The connectivity of a graph measures how many vertices must be deleted to disconnect the graph. However, since this value is based on a worst-case situation, it does not reflect how well connected the graph is in a global sense. For example, a graph $G$ obtained by adding one edge joining two large complete graphs has the same connectivity as a tree. However, it is much easier to disturb the tree, which is relevant if they both model communication systems.

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In 2002, Beineke, Oellermann and Pippert [1] introduced a parameter to give a more refined measure of the global “amount” of connectivity. The average connectivity of a graph $G$ with $n$ vertices, written $\pi(G)$, is defined to be $\sum_{u,v \in V(G)} \frac{\kappa(u,v)}{\binom{n}{2}}$, where $\kappa(u,v)$ is the minimum number of vertices whose deletion makes $v$ unreachable from $u$. By Menger’s Theorem, $\kappa(u,v)$ is equal to the maximum number of internally disjoint paths joining $u$ and $v$. Note that $\pi(G) \geq \kappa(G) = \min_{u,v \in V(G)} \kappa(u,v)$.

In Section 2, we prove a bound on the average connectivity in terms of matching number and characterize when equality holds.

In Section 3, the average edge-connectivity of a graph $G$ will be introduced. We obtain a sharp lower bound on the average edge-connectivity of a connected cubic graph with $n$ vertices. We will show that equality holds only graphs in a family. The family is also useful to find the minimum number of perfect matchings over $n$-vertex cubic graphs having a perfect matching. In fact, Lovász and Plummer [7] conjectured in the 1970s that if $G$ is a cubic graph without cut-edges, then the number of perfect matchings in $G$ should grow exponentially many with the number of vertices of $G$. Recently, the conjecture was proved by Esperet, Kardos, King, Král, and Norin [5]. In Section 4, we show that if weaken the condition “2-edge-connectedness” to “has a perfect matching”, then a cubic graph has to have at least 4 perfect matchings and there are infinitely many cubic graphs with 4 perfect matchings.

## 2 Average Connectivity and Matching Number

Regarding average connectivity, several properties are known. The following is one of them.

**Theorem 2.1.** (Dankelmann, Oellermann, 2003) [3] If $G$ has average degree $\bar{d}$ and $n$ vertices, $\bar{d} \left( \frac{\bar{d}}{n-1} \right) \leq \pi(G) \leq \bar{d}$

We prove a bound on the average connectivity in terms of matching number. The matching number of a graph $G$ is denoted by $\alpha'(G)$. We first introduce the definitions of $M$-alternating path and $M$-augmenting path.

**Definition 2.2.** Given a matching $M$, an $M$-alternating path is a path that alternates between edges in $M$ and edges not in $M$. An $M$-alternating path whose endpoints are missed by $M$ is an $M$-augmenting path.

**Theorem 2.3.** For a connected graph $G$, $\bar{\pi}(G) \leq 2\alpha'(G)$, \hspace{1cm} (1)
and this is sharp only for complete graphs with an odd number of vertices. In addition, if \( G \) is an \( n \)-vertex connected bipartite graph, then

\[
\pi(G) \leq \left( \frac{9}{8} - \frac{3n - 4}{8n(n - 8)} \right) \alpha'(G),
\]

and this is sharp only for the complete bipartite graph \( K_{q,3q-2} \), where \( q \) is a positive integer.

**Proof.** First, we show that inequality (1) holds for any connected graph \( G \). Let \( M \) be a maximum matching in \( G \), and let \( m = |M| \). Let \( S = V(G) - V(M) \), \( s = |S| \), and \( n = |V(G)| \). Note that \( n = 2m + s \).

If \( s \leq 1 \), then \( m \geq \frac{n-1}{2} \), and the bound holds since \( \overline{k}(G) \leq n - 1 \leq 2m \). Thus, we may assume that \( s \geq 2 \).

For \( uv' \in M \), put \( v \) and \( v' \) into \( T, T' \), or \( R \) as follows:

If neither \( v \) nor \( v' \) has a neighbor in \( S \), then put both in \( T \). If \( v' \) has a neighbor in \( S \) and \( v \) does not, then put \( v \in T \) and \( v' \in T' \). If both have neighbors in \( S \), put them both in \( R \). In this last case, note that if \( v \) and \( v' \) have distinct neighbors in \( S \), then \( M \) is not maximal. Hence each has exactly one neighbor in \( S \), which forms a triangle with them.

We consider three cases to obtain upper bounds on \( \kappa(u, v) \) depending on the possible locations of distinct vertices \( u \) and \( v \).

**Case 1:** \( u \in S \). First, note that \( S \) is independent. Furthermore, if \( P \) and \( P' \) are distinct internally disjoint \( u, v \)-paths, then both of them must visit \( V(M) - T \) immediately after \( u \). Since \( P \) and \( P' \) have no vertex in common, we have \( \kappa(u, v) \leq 2m - t \), where \( t = |T| \).

**Case 2:** \( u, v \in T' \). Clearly, \( \kappa(u, v) \leq n - 1 = 2m + s - 1 \)

**Case 3:** \( u \in R \cup T \). Recall that every vertex in \( R \) has exactly one neighbor in \( S \). For the vertex after \( u \) on a \( u, v \)-path, at most one vertex of \( S \) is available. Thus, if \( u \in R \cup T \) and \( v \in V(G) \), then there are at most \( 2m \) candidates to begin such a path since \( u \in V(M) \) there are \( 2m - 1 \) other vertices in \( V(M) \), and one neighbor of \( u \) is in \( S \).

Let \( t' = |T'| \). Note that by the definitons of \( T \) and \( T' \), we have \( t' \leq t \). Then, we have

\[
\overline{k}(G) \leq \frac{(2m - t) \left( \binom{s}{2} + s(n - s) \right) + (2m + s - 1) \left( \binom{t'}{2} + 2m \left( \binom{n}{2} - \binom{s}{2} - \binom{t'}{2} - s(n - s) \right) \right)}{\binom{n}{2}} \leq \frac{(2m - t) \binom{s}{2} + (2m + s - 1) \binom{t'}{2} + (2m - t) ts + 2m(\binom{n}{2} - \binom{s}{2} - \binom{t'}{2} - ts)}{\binom{n}{2}} \leq 2m + \frac{(s - 1) \binom{t'}{2} - t \binom{s}{2} - t^2 s}{\binom{n}{2}} \leq 2m - t \frac{s^2 + ts + t - 1}{n(n - 1)} \leq 2m. \quad (3)
\]

The last inequality of (3) holds because when \( t \geq 1 \), we have \( s^2 + 3ts + t - 1 \geq 0 \) and when \( t = 0 \), we have \( t^2 s^2 + 3ts + t - 1 = 0 \).
To have equality in the last inequality of (3), we need to have \( t = 0 \) or \( t = 1 \) since if \( t \geq 2 \), then \( t^{s+3s+t-1} > 0 \).

When \( t = 1 \), equality requires \( s = 0 \), which implies that \( M \) is a perfect matching. Thus \( 2m = n \). In this case, \( \pi(G) \leq n - 1 < n = 2m \), which implies that we cannot have equality in (3).

If \( t = 0 \) and \( s \geq 2 \), then we have a bigger matching than \( M \), since every vertex in \( R \) has exactly one neighbor in \( S \) and \( G \) is connected. Thus, when \( t = 0 \), equality requires \( s \leq 1 \).

If \( s = 0 \), then \( 2m = n \), which is bigger than \( n - 1 \geq \pi(G) \). In this case, we cannot have equality in (3). Thus, we have \( s = 1 \), which implies \( 2m = n - 1 \). Then \( n \) is odd, and if \( \pi(G) = n - 1 \), then \( G = K_n \). Thus, equality holds only when \( G \) is a complete graph with an odd number of vertices.

To prove that inequality (2) holds, we consider an \( n \)-vertex connected bipartite graph \( G \) with partite sets \( A \) and \( B \). Let \( M \) be a maximum matching in \( G \). Let \( m = |M| \), let \( A_1 = A - V(M) \), and let \( B_1 = B - V(M) \). Let \( B_2 \) be all vertices in \( B \) that are reachable by an \( M \)-alternating path from a vertex in \( A_1 \), and let \( A_2 \) be all vertices in \( A \) that are reachable by an \( M \)-alternating path from a vertex in \( B_1 \). Note that \( A_1 \cap A_2 = \emptyset \) and \( B_1 \cap B_2 = \emptyset \) and there are no edges of \( M \) joining \( A_2 \) and \( B_2 \); otherwise we have a bigger matching than \( M \) by making a \( M \)-augmenting path, which is a contradiction.

Let \( A_3 = A - (A_1 \cup A_2) \) and \( B_3 = B - (B_1 \cup B_2) \). Let \( |A_i| = a_i \) and \( |B_i| = b_i \) for \( i = 1, 2, 3 \).

We consider five cases to obtain lower bounds on \( \kappa(u, v) \) depending on the possible locations of distinct vertices \( u \) and \( v \).

**Case 1:** \( u, v \in A_2 \), or \( u, v \in B_2 \).

If \( u, v \in A_2 \), then since every \( u, v \)-path must pass through a vertex in \( B \), we have \( \kappa(u, v) \leq b = m + b_1 \).

If \( u, v \in B_2 \), then we replace \( B \) and \( b = m + b_1 \) in the proof of the case \( u, v \in A \) by \( A \) and \( a = m + a_1 \).

**Case 2:** \( u \in A \) and \( v \in (A - A_2) \), or \( u \in B \) and \( v \in (B - B_2) \).

If \( u \in A \) and \( v \in (A - A_2) \), then since every \( u, v \)-path must pass through at least one vertex in \( B \cap V(M) \), we have \( \kappa(u, v) \leq m \).

If \( u \in B \) and \( v \in (B - B_2) \), then we replace \( B \cap V(M) \) in the proof of the case \( u \in A \) and \( v \in (A - A_2) \) by \( A \cap V(M) \).

**Case 3:** \( u \in A - A_2 \) and \( v \in B - B_2 \).

Every \( u, v \)-path must pass through at least two vertices in \( M \) except the path of length one \( uv \), which implies that \( \kappa(u, v) \leq m \).
Thus, we have
\[ \pi(G) \leq \frac{m{\binom{n}{2}} + a_1(\binom{b_2}{2}) + b_1(\binom{a_2}{2})}{\binom{n}{2}} = m + \frac{a_1(\binom{b_2}{2}) + b_1(\binom{a_2}{2})}{\binom{n}{2}} \leq m + \frac{(a_1 + b_1)(\binom{b_2 + a_2}{2})}{\binom{n}{2}}. \] (4)

Since no edge of \( M \) joins \( A_2 \) to \( B_2 \), all vertices of \( A_2 \) match into \( B_3 \) under \( M \). Thus, we have \( a_2 \leq b_3 \). Similarly, we have \( b_2 \leq a_3 \). Thus, we have \((a_1 + b_1) + 2(a_2 + b_2) \leq n \) and \((a_2 + b_2) \leq m \). Since \( n - 2 \geq (a_1 + b_1) + 2(a_2 + b_2 - 1) \geq 2\sqrt{2(a_1 + b_1)(a_2 + b_2 - 1)} \), we have \((a_1 + b_1)(a_2 + b_2 - 1) \leq \frac{(n - 2)^2}{8} \). Thus, we have
\[ \pi(G) \leq m + \frac{(a_1 + b_1)(\binom{b_2 + a_2}{2})}{\binom{n}{2}} \leq m + \frac{(a_1 + b_1)(a_2 + b_2 - 1)}{n(n-1)}(a_2 + b_2) \]
\[ \leq m + \frac{(n - 2)^2}{8n(n-1)}m = \frac{9}{8}m - \frac{3n - 4}{8n^2 - 8n}m. \]

To have equality in the last inequality of (4), \( a_1 = 0, b_2 = 0 \) or \( b_1 = 0, a_2 = 0 \). We may assume that \( a_1 = 0 \) and \( b_2 = 0 \). To have equality in those inequalities \( a_2 \leq b_3 \) and \( b_2 \leq a_3 \) below 4, \( a_2 = b_3 \) and \( b_2 = a_3 \), which implies that \( a_3 = 0 \). Similarly, we have \( b_1 + 2a_2 = n \) and \( \sqrt{b_1} = \sqrt{2(a_2 - 1)} \) from the inequalities \((a_1 + b_1) + 2(a_2 + b_2) \leq n \) and \( n - 2 \geq (a_1 + b_1) + 2(a_2 + b_2 - 1) \geq 2\sqrt{2(a_1 + b_1)(a_2 + b_2 - 1)} \) below 4. Therefore, we have \( n = a_2 + b_1 + b_2 = a_2 + 2a_2 - 2 + a_2 = 4a_2 - 2 \). To have equality in the last inequality of (4), every two vertices \( u \) and \( v \) in \( B \) must have \( \kappa(u, v) = m \). Thus every vertex in \( B \) must be adjacent to all the vertices in \( A_1 \), which implies that only the complete bipartite graph \( K_{a_2,3a_2-2} \) satisfies the equality. Thus equality holds only for \( K_{q,3q-2} \) for a positive integer \( q \).

3 Average Edge-connectivity in Regular Graphs

In this section, we introduce a concept in terms of edge-connectivity analogously to average connectivity. A graph \( G \) is \( k \)-edge-connected if every subgraph obtained by deleting fewer than \( k \) edges is connected; the edge-connectivity of \( G \), written \( \kappa'(G) \), is the maximum \( k \) such that \( G \) is \( k \)-edge-connected. The average edge-connectivity of a graph \( G \) with \( n \) vertices, written \( \overline{\kappa}'(G) \), is defined to be \( \sum_{u,v \in V(G)} \kappa'(u, v)/\binom{n}{2} \), where \( \kappa'(u, v) \) is the minimum number of edges whose deletion makes \( v \) unreachable from \( u \), which is same as the number of edge-disjoint paths between \( u, v \). Note that \( \overline{\kappa}'(G) \geq \kappa'(G) = \min_{u,v \in V(G)} \kappa'(u, v) \).

This new parameter shares certain properties with the average connectivity. Even if we replace \( \pi(G) \) in Theorem 2.1 and Theorem 3.2 by \( \overline{\kappa}'(G) \), then the inequalities hold.

**Theorem 3.1.** If \( G \) has average degree \( d \), and \( |V(G)| = n \),

\[ \overline{\kappa}'(G) \leq \frac{m}{\binom{n}{2}} \leq \frac{m + \frac{(a_1 + b_1)(a_2 + b_2 - 1)}{n(n-1)}(a_2 + b_2)}{\binom{n}{2}}. \] (4)
Theorem 3.2. For a connected graph $G$,
\[
\overline{\kappa'}(G) \leq 2\alpha'(G),
\] (5)
and this is sharp only for complete graphs with an odd number of vertices. In addition, if $G$ is an $n$-vertex connected bipartite graph, then
\[
\overline{\kappa'}(G) \leq \left( \frac{9}{8} - \frac{3n - 4}{8n(n - 8)} \right) \alpha'(G),
\] (6)
and this is sharp only for the complete bipartite graph $K_{q,3q-2}$, where $q$ is a positive integer.

The proof is the same as in Theorem 3.2 if we look at the second vertex in sets of edge-disjoint paths.

However, consider the graph $G_1$ in figure 1, which is the graph obtained from $P_{s+1}$ by replacing an edge with a copy of $K_q$. Two successive copies of $K_q$ share one vertex. The total number of vertices is $1 + s(q - 1)$. Since $\pi(G_1) = 1 + O(\frac{q}{s})$ and $\overline{\kappa'}(G_1) = q - 1$, we have a big difference between the average connectivity and the average edge-connectivity of this graph.

In order to analyze average edge-connectivity of regular graphs, we first introduce a notion of $i$-balloon of a regular graph. If a regular graph $G$ has a cut-edge, then we get components after we delete all cut-edges of $G$. We define an $i$-balloon in $G$ to be such a component incident to $i$ cut-edges. Note that an 1-balloon in $G$ is a balloon in $G$ introduced in [9], where a balloon of a graph is a maximal 2-edge-connected subgraph, which is incident to exactly one cut-edge of $G$, and for any $i \geq 1$, an $i$-balloon of $G$ is a maximal 2-edge-connected subgraph of $G$ except when it is a single vertex, and the resulting graph obtained by shrinking each $i$-balloon to a single vertex is a tree. For a cubic graph, its smallest 1-balloon is the smallest possible balloon in a cubic graph, denoted by $B_1$ in [9]. The smallest 2-balloon in a cubic graph is $K_4 - e$. We denote the smallest $i$-balloon in a cubic graph by $B_i$.

Now we compute the average edge-connectivity of several cubic graphs with $n$ vertices less than 10. Before doing it, we first give a lemma.
Lemma 3.3. If $G$ has a vertex subset $S$ in $V(G)$ such that $|S, \bar{S}| < \delta(G)$, then $|S| \geq \delta(G) + 1$. Furthermore, if $G$ is a $(2r + 1)$-regular graph and $S$ is a vertex subset in $V(G)$ such that $|S, \bar{S}| < 2r + 1$, then $|S, \bar{S}| \equiv |S| \mod 2$.

Proof. If $|S| \leq \delta(G)$, then $|S, \bar{S}| \geq |S|(|\delta(G) - (|S| - 1)| \geq \delta(G)$. Thus the first statement is true.

For the second statement, $(2r + 1)|S| - |S, \bar{S}|$ is even by the Degree-Sum Formula. Thus, we have $|S, \bar{S}| \equiv (2r + 1)|S| \equiv |S| \mod 2$. □

By the Degree-Sum Formula, an odd regular graph has even number of vertices. If $n = 4$, then it is $K_4$. Since the edge-connectivity of $K_4$ is $3$, we have $\frac{\kappa^r(K_4)}{2} = 3{\binom{4}{2}}$, which is smaller than $\frac{7 \times 4 + 58}{4}$. If $n = 6$, then $\kappa(G) = 3$ since if its edge-connectivity is less than $3$, then it has to have at least $8$ vertices by Lemma 3.3. Hence the average edge-connectivity of a cubic graph with $6$ vertices, $3$, is greater than $\frac{7 \times 6 + 58}{4}$. If $n = 8$, then its edge-connectivity is at least $2$ since if its edge-connectivity is equal to $1$, then it has to have at least $10$ vertices by Lemma 3.3. If its edge-connectivity is equal to $2$, then it is the graph obtained by adding two edges between two $B_i$’s. Its edge-connectivity is $2{\binom{8}{2}} + 2$, and note that $3{\binom{8}{2}} \geq 2{\binom{8}{2}} + 2$. Thus the average edge-connectivity of a cubic graph with $8$ vertices is greater than $\frac{7 \times 8 + 58}{4}$. Now, we prove that every cubic graph other than $K_4$ satisfies the bound in the following theorem.

Theorem 3.4. If $G$ is a connected cubic graph $G$ with $n$ vertices, which is not $K_4$, then

$$\frac{\kappa^r(G)}{2} \geq \frac{n}{2} + \frac{7n + 58}{4}.$$ 

Proof. Consider a minimal counterexample $G$ with $n$ vertices.

Claim 1: The edge-connectivity of $G$ is $1$. If not, then $\frac{\kappa^r(G)}{2} \geq \frac{n}{2} \geq \frac{n}{2} + \frac{7n + 58}{4}$ for $n \geq 10$. In the above, we showed that a cubic graph with vertices less than $10$ satisfies the bound when it is not $K_4$.

Claim 2: Every $1$-balloon of $G$ is $B_1$. If $D_1$ is an $1$-balloon of $G$ with $D_1 \neq B_1$ and $|V(D_1)| = 5 + a$, then Degree-Sum formula guarantees that $a$ is an even positive integer, which implies that $a \geq 2$. Let $G'$ be the graph obtained from $G$ by replacing $D$ with $B_1$. Note that $G'$ is cubic. In addition, since $G'$ also has a cut-edge, $10 \leq n - a = |V(G')| \leq |V(G)|$. Since $G$ is larger graph than $G'$, which is not $K_4$, by the hypothesis of $G$, we have $\kappa^r(G')(\frac{n - a}{2}) \geq \frac{7}{4}(n - a) + \frac{29}{2}$. By the construction of $G'$ and the fact that $B_1$ is $2$-edge-connected,

$$\frac{\kappa^r(G)}{2} = \frac{\kappa^r(G')}{2} - \frac{\kappa^r(B_1)}{2} - 5(n - a - 5) + \frac{\kappa^r(B_1)}{2} + (5 + a)(n - a - 5) \geq \frac{(n - a)}{2} + \frac{7}{4}(n - a) + \frac{29}{2} - 26 - 5(n - a - 5) + \frac{2(5 + a)}{2} + (5 + a)(n - a - 5).$$
\[ \binom{n}{2} + \frac{a^2 + a - 2an}{2} + \frac{7}{4}n - \frac{7a}{4} + \frac{29}{2} - 26 + (5 + a)(5 + a - 1) + a(n - a - 5) \]

\[ = \binom{n}{2} + \frac{7}{4}n + \frac{2a^2 + 11a + 34}{4} > \binom{n}{2} + \frac{7}{4}n + \frac{29}{2} \]

for \( a \geq 2 \), which is a contradiction to the assumption that \( G \) is a counterexample.

Claim 3: Every 2-balloon of \( G \) is \( B'_1 \). If \( D_2 \) is a 2-balloon of \( G \) with \( D_2 \neq B'_1 \) and \(|D_2| = 4+a\), then Degree-Sum formula guarantees that \( a \) is an even positive integer, which implies that \( a \geq 2 \). Let \( G' \) be the graph obtained from \( G \) by replacing \( D_2 \) with \( B'_1 \) in order that \( G' \) is a cubic graph. Thus, \( G' \) has \( n-a \) vertices for \( a \geq 2 \), and by the hypothesis that \( G \) is a minimal counterexample, \( \overline{\kappa}(G'(\binom{n}{2} - a)) \geq \left( \binom{n-a}{2} - \frac{7}{4}(n-a) + \frac{29}{2} \right) \). By the construction of \( G' \), and the fact that \( B'_1 \) is 2-edge-connected, we have

\[ \overline{\kappa}(G') \left( \binom{n}{2} - a \right) = \overline{\kappa}(G') \left( \binom{n-a}{2} \right) - \overline{\kappa}(B'_1) \left( \binom{4}{2} \right) - 4(n-a-4) + \overline{\kappa}(B'_1) \left( \binom{4+a}{2} \right) + (4+a)(n-a-4) \]

\[ \geq \binom{n-a}{2} + \frac{7}{4}(n-a) + \frac{29}{2} - 13 - 4(n-a-4) + 2(4+a)(n-a-4) \]

\[ = \binom{n}{2} + \frac{a^2 + a - 2an}{2} + \frac{7}{4}n - \frac{7a}{4} + \frac{29}{2} - 13 + a(n-a-4) + (4+a)(n-a-1) \]

\[ = \binom{n}{2} + \frac{7}{4}n + \frac{29}{2} + \frac{2a^2 + 7a + 48}{4} > \binom{n}{2} + \frac{7}{4}n + \frac{29}{2} \]

for \( a \geq 2 \), which is a contradiction to the assumption that \( G \) is a counterexample.

Claim 4: \( G \) has no \( k \)-balloons for \( k \geq 3 \). Assume that \( G \) has a \( k \)-balloon for \( k \geq 3 \). Since \( G \) contains \( B_1 \) as an induced subgraph by Claim 1, choose a \( k \)-balloon \( D_k \) for \( k \geq 3 \) which is closest to \( B_1 \). \( D_k \) is incident to \( B_1 \)'s or \( B_1 \) by the choice of \( D_k \). If \( k \geq 4 \), then \(|V(D_k)| \geq k \) since each vertex in \( V(D_k) \) is incident to at most one cut-edge. Thus, we can assume that \(|V(D_k)| = k + a \) with \( a \geq 0 \). Suppose that there are \( m \) \( B_k \)s between \( B_1 \) and \( D_k \). Let \( G' \) be the graph obtained from \( G \) by deleting all \( B_k \)s between \( D_k \) and \( B_1 \), deleting \( B_1 \), and replacing \( D_k \) with \( C_{k-1} \) and attaching each cut-edge except one between \( D_k \) and \( B_k \) to each vertex in \( C_{k-1} \). Note that \( G' \) has \( n-a-4m-6 \) vertices. Thus, we have \( \overline{\kappa}(G'(\binom{n}{2} - 4m - 6)) \geq \binom{n-a-4m-6}{2} + \frac{7}{4}(n-a-4m-6) + \frac{29}{2} \). The construction of \( G' \) guarantees that

\[ \overline{\kappa}(G) \left( \binom{n}{2} \right) = \overline{\kappa}(G') \left( \binom{n-a-4m-6}{2} \right) - \overline{\kappa}(C_{k-1}) \left( \binom{k-1}{2} \right) \]

\[ -(k-1)(n-a-4m-k-5) + \binom{4m+5+a+k}{2} + (\overline{\kappa}(D_k) - 1) \binom{k+a}{2} \]

\[ + m(\overline{\kappa}(B_4) - 1) \binom{4}{2} + (\overline{\kappa}(B_5) - 1) \binom{5}{2} + (4m+5+a+k)(n-4m-a-k-5) \]

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A graph consists of only \( k \) containing no tree with maximum degree 2, which is a path. Thus, which is a contradiction to the assumption that \( G \) is a counterexample. Thus, we can assume that \( k \) is equal to exactly 3. Let \( |V(D_3)| = a \). Note that \( a \geq 1 \). Assume that there are \( m \) \( B_3 \)s between \( B_1 \) and \( D_3 \). Let \( G' \) be the graph obtained from \( G \) by deleting all \( B_3 \)s between \( D_3 \) and \( B_1 \), deleting \( B_1 \), replacing \( D_3 \) with \( B_4 \), and attaching each of two remaining cutedges to vertices of \( B_4 \) with degree 2. Note that \( G' \) has \( n-a-4m-1 \) vertices. Since \( G' \) is smaller than \( G \), we have \( \overline{\kappa}(G') \left( \frac{n-a-4m-a}{2} \right) \geq \left( \frac{n-a-4m-1}{2} \right) + \frac{7}{4}(n-a-4m-1) + \frac{29}{2}. \)

By the construction of \( G' \), we have

\[
\overline{\kappa}(G') \left( \frac{n}{2} \right) = \overline{\kappa}(G') \left( \frac{n-a-4m-1}{2} \right) - \overline{\kappa}(B_4) \left( \frac{4}{2} \right) - (4)(n-a-4m-5) + \left( \frac{a+4m+5}{2} \right) + (\overline{\kappa}(D)-1) \left( \frac{a}{2} \right) + m(\overline{\kappa}(B_4)-1) \left( \frac{4}{2} \right) + (\overline{\kappa}(B_5)-1) \left( \frac{5}{2} \right) + (a+4m+5)(n-a-4m-5)
\]

\[
\geq \left( \frac{n-a-4m-1}{2} \right) + \frac{7}{4}(n-a-4m-1) + \frac{29}{2} - 13 - 4(n-a-4m-5) + \left( \frac{a+4m+5}{2} \right) + \left( \frac{a}{2} \right)
\]

\[
+ 7m + 16 + (a+4m+5)(n-a-4m-5) = \left( \frac{n}{2} \right) + \frac{7}{4}n + \frac{1}{2}(a-\frac{9}{4})^2 + 615 \left( \frac{32}{32} \right) + 4k - \frac{87}{4} + 4k
\]

\[
= \left( \frac{n}{2} \right) + \frac{7}{4}n + \frac{1}{2}(a-\frac{9}{4})^2 + \frac{615}{32} + 4k > \left( \frac{n}{2} \right) + \frac{7}{4}n + \frac{29}{2},
\]

which is a contradiction to the assumption that \( G \) is a counterexample. Therefore, \( G \) contains no \( k \)-balloons for \( k \geq 3 \). By the above claims, the minimal counter example should be a graph consists of only \( B_4 \) and \( B_5 \). After contracting each \( k \)-balloon of \( G \), we should get tree with maximum degree 2, which is a path. Thus \( G \) should be two \( B_5 \) on the endvertices and \( B_3 \)s are attached each other. But in that case, \( \overline{\kappa}(G) = \left( \frac{n}{2} \right) + \frac{7}{4}n + \frac{29}{2} \), which satisfy the proposition.

Thus, it contradicts the assumption that \( G \) is a counterexample. And the inequality is sharp only for \( n \equiv 2 \) \( \text{mod} \, 4 \) by the above example.

Figure 2 describes an infinite family of graphs for which equality holds in Theorem 3.4. More generally, we conjecture for \((2r+1)\)-regular graphs.
Conjecture 3.5. If $G$ is a connected $(2r + 1)$-regular graph $G$ with $n$ vertices, then
\[
\kappa'(G) \left(\frac{n}{2}\right) \geq \min\{2 \binom{n}{2}, \binom{n}{2} + \frac{(r - 2)(r^2 + 2r - 1)}{2(r + 1)} n + \frac{r^3 + 4r^2 + r - 8}{r + 1}\}.
\]

If the above conjecture holds, then it is sharp for infinitely many $n$. Let $A = P_3 + rK_2$ and let $B = K_{2r+2} - e$. Note that $A$ is the smallest possible 1-balloon and $B$ is the smallest possible 2-balloon in a $(2r + 1)$-regular graph. Consider a path $P$ with length at least 1. Replace both end-vertices of $P$ by $A$ and the other vertices of $P$ by $B$. We define a $(2r + 1)$-chain to be the resulting graph. A $(2r + 1)$-chain is $(2r + 1)$-regular and satisfies the equality in the Conjecture 3.5. Another usage of the graphs $A$ and $B$ is to find the minimum matching number over $n$-vertex $(l$-edge-)connected $k$-regular graphs and a relationship between eigenvalues and matching number in regular graphs. (See [9], [10], [2])

4 Perfect Matchings in Regular Graphs

The graphs achieving equality in Theorem 3.4 are also helpful to find a lower bound for the number of perfect matchings over $n$-vertex connected cubic graphs.

In 1892, Petersen [12] proved that every cubic graph without cut-edges has a perfect matching. Thus, it is natural to ask how many perfect matchings a 2-edge-connected cubic graph must have. In 1970s, Lovász and Plummer [7] conjectured that every 2-edge-connected cubic graph with $n$ vertices has at least exponentially many (in $n$) perfect matchings. The first result on the conjecture is the following theorem proved by Edmonds, Lovász, and Pulleyblank [6].

Theorem 4.1. (Edmonds, Lovász and Pulleyblank [6]; Naddef [8]) If $G$ is an $n$-vertex connected cubic graph without cut-edges, then $m_p(G) \geq \frac{n}{4} + 2$, where $m_p(G)$ denotes the number of perfect matchings in $G$.

Recently, the conjecture was proved by Esperet, Kardos, King, Král, and Norin [5]. With the result, they also proved more more generally that every $(k - 1)$-edge-connected $k$-regular graph with $n$ vertices has at least exponentially many perfect matchings.

A $k$-regular graph $G$ with $\kappa'(G) \geq k - 1$ has to have a perfect matching. If we weaken the condition “$(k - 1)$-edge-connectedness” to “has a perfect matching”, then how many perfect matchings must a $k$-regular graph have? In this section, we answer this question for $k = 3$. 

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We will use Pléšnik’s Theorem, which states that if $G'$ is the graph obtained from a $(k - 1)$-edge-connected $k$-regular multigraph $G$ by deleting at most $k - 1$ edges in $G$, then $G'$ has a perfect matching.

**Lemma 4.2.** If $B$ is a balloon with the neck $v$ in a cubic graph, where the neck of a balloon is the vertex with degree 2, then there are at least two near perfect matchings not using $v$.

**Proof.** Let $x$ and $y$ be the two vertices adjacent to $v$ in $B$. Let $B'$ be the resulting graph obtained from $B$ by adding an edge between $x$ and $y$ after deleting the vertex $v$. Note that $B'$ is a cubic multigraph without cut-edges. By Pléšnik’s Theorem, there are at least two perfect matchings not using the added edge, which implies that there are at least two near perfect matchings in $B$. \qed

**Theorem 4.3.** Every $n$-vertex connected cubic graph with a perfect matching except for $K_4$ has at least four perfect matchings. In addition, equality holds for all 3-chains.

**Proof.** Assume that $G$ is a connected cubic graph with $n$ vertices other than $K_4$. Note that $n \geq 6$. If $G$ has no cut-edges, then by Theorem 4.1, $G$ has at least four perfect matchings.

Now, assume that $G$ has a cut-edge. Hence we have at least two balloons (See Lemma 3.3 in [9]). By Lemma 4.2, each balloon has at least two near-perfect matchings not using its neck. Since there are are at least two balloons, we have at least four perfect matchings.

Consider a 3-chain $G$. There are exactly two near perfect matchings not using the neck of each copy of $B_1$ in $G$, and since every perfect matching in $G$ has to use all cut-edges in $G$, we have only one choice in each copy of $K_4 - e$. Thus, we have exactly four perfect matchings in $G$. \qed

We believe that if for $k \geq 4$, $G$ is a connected $k$-regular graph with $n$ vertices, then $m_p(G)$ (in $n$) is at least exponentially many. In particular, we conjecture the following for odd regular graphs.

**Conjecture 4.4.** If for $r \geq 1$, $G$ is a connected $(2r + 1)$-regular graph with $n$ vertices, then for some constant $c$, $m_p(G) \geq c(2r - 1)!! \frac{n - 2(2r + 3)}{2r + 2}$.\n
Note that the conjecture 4.4 is true when $r = 1$. In fact, when $r = 1$, the constant number $c$ is equal to 4.

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References


