Claim: $C$ has a $g_3^3$ canonical curve has a trisection line.

If $P, Q, R \in C$, $P, Q, R$ colinear $\Rightarrow$

$$\dim I_{K-P-Q-R} = \dim I_{K-L} - 2 = g - 3$$

Let $\dim I_{L+P+R} = \dim I_{K-P-Q-R} + 3 + 1 - g$

So colin. $\Rightarrow \dim I_{L+P+R} = 1, g_3^3$

Remark: more generally, a nonhyperelliptic curve has a $g_3^3$ canonical curve by a $d$-seceent $d-r-1$ plane

Ex: Complete intersection of three degree 2 hypersurfaces, degree $2^3 = 8$ curve in $\mathbb{P}^9$

Adjunction $\omega_C \cong \omega_{\mathbb{P}^4}(-21, 22 | 1) \cong \omega_C(1)$

So $C$ is canonical, $g = 5$

Now, $C$ has no $g_3^3$: $\text{If } L \text{ met } C \text{ 3 times,}$

$C : \omega_{\mathbb{P}^4}(-22, -2)$ Then $L$ meets $\omega_{\mathbb{P}^4}$ 3 times

$\deg \omega = 2 \Rightarrow L \cdot \omega = L \cdot C \cdot C$, contradiction.

A $g_3^3$ can be constructed geometrically (see text)
What if $C$ has $g \geq 5$?

As the $g_i$'s move in its pencil, the trisecant lines sweep out a "ruled surface" $S \supset C$.

By using $0 \to I_c(2) \to O_{pp}(2) \to O_c(2) \to 0$.

It can be checked that $\text{dim } H^2(I_c(2)) > 3$, and in fact $= 3$.

The intersection of these quadrics containing the canonical curve can be

\[ Q \cap Q' = \{ C \mid C \text{ has } g \geq 5 \}. \]

In the latter case, an additional degree 3 hypersurface are needed to define $C$.

Petri determined the possible ideals of all canonical curves. They are always defined by degree 2 hypersurfaces, unless $C$ has a $g_i$.

- $C$ has a $g_i$.
- $C$ has $g \geq 6$ and is isomorphic to a plane curve of degree 5.
Suppose \( C \) is a curve of degree 5.

\[ U_C = \mathcal{O}_{\mathbb{P}^2}(5) \cong \mathcal{O}(2) \]

So canonical map is composed

\[ C \to \mathbb{P}^2 \to \mathbb{P}^5 \]

Let \( \text{Sel}^{\mathbb{P}^5} \) be the Veronese surface, image of \( \mathbb{P}^2 \).

Let \( \mathcal{Q} \in H^0(I_C(2)) \).

Claim: \( \mathbb{P}^2 \subset Z(\mathcal{Q}) \).

If not, then \( Z(\mathcal{Q}) \cap \mathbb{P}^2 \) is a curve of degree 4 containing \( C \), contr. \( \text{This is a separate line of thought.} \)

Thus, \( \mathbb{P}^2 \subset \cap Z(\mathcal{Q}) \).

In fact, equal.

Again, need degree 3 hypersurface to define \( C \).
Classification of curves in $\mathbb{P}^3$ (unsolved problem)

Equivalently, every curve $C$ in $\mathbb{P}^3$ has a non-special very ample divisor $D$.

Prop. C, genus $g \geq 2$, has non-special very ample $D$ if $\deg D \geq d \geq g + 3$.

Case I. $C \subset \mathbb{P}^3$, degree $d$, genus $g$ with non-special hyperplane section $\iff$ et cetera.

1) $g = 0$, $d \geq 1$
2) $g = 1$, $d \geq 3$
3) $g \geq 2$, $d \geq g + 3$

Pf. $D$ hyperplane section $\iff D$ is very ample $\iff$
case 1. $\dim \mathcal{I}_{C, d - 2} \geq d - 2g + 1$, and in case 2
by $\deg D$. This gives $C \hookrightarrow \mathbb{P}^N$.

case 1: $\deg D + 1$ ample, with

$$N = \begin{cases} \mathcal{O} & \text{if } g = 0, 1, 2, \ldots, \infty \in \mathbb{P}^N \\ N_2^3 & \text{if } g \geq 2 \end{cases}$$

If $N \leq 3$, done. If $N > 3$, project $C$. 
pf of proposition

D nonspecial, very c-s, any D = d = 1 \text{ dim } 10\lambda 1 = d - g - 3

C \subseteq P^{d-g-3}

g \geq 2 = C \times P^1 = d - g \geq d - g - 2

If d - g - 2, then \mu_c = \Omega(d - g) = 0

\text{as result} \Rightarrow d \leq 3 \Rightarrow g \leq 1, \text{ contradiction}

\text{Proof: } C \subseteq P^3 \text{ nondegenerate, hyp secant } 0 \text{ space for } d \geq 6 + g \geq \frac{d}{2} + 11. \text{ Furthermore, the only such curve with } d = 6 \text{ is the canonical curve of } g = 4

pf: \text{ D special } \Rightarrow \text{ dim } 10\lambda 1 \leq \frac{d}{4}
\text{ dim } 10\lambda 1 \geq 3 = 1 \text{ d} \geq 6

D special \Rightarrow d \geq 2g - 2 = g \geq \frac{d}{2} + 11

\text{ If } d = 6 \text{ then dim } 10\lambda 1 - 3 = 0 \text{ (Chiaha)}

D = 0 (\in) D = K, \omega = \text{ C hyperbolic}, D = 0 \Rightarrow

\text{ show that curve is cut out by divisors and hyperbolic points,}

\text{ construction.}
The Castelnuovo bound. C(CP^2) d \geq g \text{ genera } \text{ non-degenerate. Then } d \geq 3 \text{ ann } \\
\frac{1}{4} \int d^2 \text{ done} \\
g \leq \left\lfloor \frac{1}{2} (2d-1) - 1 \right\rfloor \\
\text{done} \\
\text{If equality, then C(C) quasi such}

\text{Examples of equality: } \begin{align*}
d &= 8 \quad g = 9 \\
0(C) &= (4,4) \\
d &= 11 \quad g = 3 \text{.3} \\
d &= 7 \quad g = 6 \\
& \text{and } C(4,3) \text{ or quasi } \\
&d = 9, \quad g = 3.2 \text{.6}
\end{align*}

Aside: what is a curve of degree when one (d, g) the degree and genus of a curve on a quadric?

\text{Fl}(a,b) s.t. \begin{align*}
d &= a \times b \quad g &= (a-1)(b-1) \\
\text{Note: } (d-2)^2 - 4g &= (a-b)^2 \\
\text{Commonly, if } (d-2)^2 - 4g \text{ is a perfect square, can solve for } a, b.
Lemma: If \( C \subseteq \mathbb{P}^n \) is non-degenerate, then for any \( 1 \leq j \leq n \), there exists \( d \) distinct points, no three collinear.

Proof. See text + HW

Proof of Castelnuovo bound:

Choose hyperplane section \( D \). Fix \( 1 \leq j \leq n \).

Claim: \( \dim \text{Ind}(\cdot) - \dim \text{Ind}(\cdot) \geq \min \left( d, \frac{2n+1}{2} \right) \)

Assume claim. Write \( \alpha \in \left[ \frac{d-1}{2} \right] \). Then

\[
\dim \text{Ind}(\cdot) - \left( \dim \text{Ind}(\cdot) - \dim \text{Ind}(\cdot) \right) + \left( \dim \text{Ind}(\cdot) - \dim \text{Ind}(\cdot) \right)
\]

\[
\geq \left( n-\alpha \right) d + \left( \frac{2n+1}{2} \right) + \cdot + \cdot + \cdot
\]

\[
= \left( n-\alpha \right) d + \frac{1}{2} \cdot \cdot \cdot
\]

Also, \( \dim \text{Ind}(\cdot) = n-\gamma \). For \( n \gg 0 \)

\[
\Rightarrow \quad \gamma \leq \alpha d - \frac{1}{2} \cdot \cdot \cdot
\]

\[
d \cong \gamma \Rightarrow \quad \frac{d}{2} = \gamma \leq \frac{1}{2} \cdot \cdot \cdot
\]

\[
d \leq \gamma \Rightarrow \quad \frac{d}{2} = \gamma \leq \frac{1}{2} \cdot \cdot \cdot
\]
If equality is attained, then all inequalities are equalities. In particular,

$$\dim (1201) \leq 3 \times 8 = 24$$

$$\implies \dim (\mathcal{O}(1201)) \leq 24.$$ 

$$\mathcal{O} \to J_{21} \to \mathcal{O}(21) \to \mathcal{O}(2) \to 0$$

$$0 \to H^2(\mathcal{O}(21)) \to H^2(\mathcal{O}(2)) \to H^2(\mathcal{O}_2) \to 0$$

$$\dim_1 = 10 \quad \dim_2 = 5$$

$$\implies \dim H^2(\mathcal{O}(21)) \geq 5.$$ 

Remains to prove claim.

For each $i = 1, 2$, $\min (d_i, 2n+1)$, choose a sequence

$H_i$ containing $\mathcal{I}_i$ first and then $\mathcal{J}_i$.

until all $H_1, H_2$ contain $\mathcal{F}_1, \mathcal{F}_2$.

then take remaining points $u_1, \ldots, u_t$ to $\mathcal{O}$.

which do not contain any $\mathcal{F}_i$.

Then $H_{11} + H_{22} \subset \mathcal{I}_0 - \mathcal{F}_1 - \mathcal{F}_2 - \cdots - \mathcal{F}_t$.

but does not contain $\mathcal{F}_i$.

$H_{11} \cup H_{22}$ contains a free point of $\mathcal{I}_0 - \mathcal{F}_1 - \cdots - \mathcal{F}_t$.

$$\dim (\mathcal{I}_0 - \mathcal{F}_1 - \cdots - \mathcal{F}_t) = \dim (\mathcal{I}_0 - \mathcal{F}_1 - \cdots - \mathcal{F}_t) \leq \dim (\mathcal{I}_0 - \min (d_i, 2n+1))$$

$$\dim (\mathcal{I}_0 - \mathcal{F}_i)$$