1. Exercise 10.7, modular invariance of $Z_{E_6,A_{p-1}}$.

(a) We have

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}$$

so with $p' = 12$ we have

$$h_{7,s} - h_{1,s} = \frac{(7p - 12s)^2 - (p - 12s)^2}{48p} = p - 3s,$$

$$h_{8,s} - h_{4,s} = \frac{(8p - 12s)^2 - (4p - 12s)^2}{48p} = p - 2s,$$

$$h_{11,s} - h_{5,s} = \frac{(11p - 12s)^2 - (5p - 12s)^2}{48p} = 2p - 3s.$$

Since these are all integers, each of the three block characters $C_{1,s} = \chi_{1,s} + \chi_{7,s}$, $C_{4,s} = \chi_{4,s} + \chi_{8,s}$, $C_{5,s} = \chi_{5,s} + \chi_{11,s}$ transform by the same phase, leading immediately to $T$-invariance.

(b) We have that $1 \leq s \leq p - 1$. Since $h_{r,s} = h_{p-r,12-s}$ we conclude that

$$C_{1,s} = C_{5,12-s}, \quad C_{4,s} = C_{4,12-s}$$

and there are only half as many independent $C_{r,s}$, i.e. $3(p-1)/2$ of them. From the $S$ transformation

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{1}{6p}}(-1)^{1+s_{p+r}\sigma} \sin\left(\frac{\pi pr\rho}{12}\right) \sin\left(\frac{12\pi s_{p}\sigma}{p}\right)$$

we see that

$$C_{1,s}(-1/\tau) = 2\sqrt{\frac{1}{6p}} \sum_{r,\sigma} (-1)^{1+s_{p+r}\sigma}. $$
\[
\left( \sin \left( \frac{\pi \rho}{12} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) + \sin \left( \frac{7\pi \rho}{12} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) \right) \chi_{\rho\sigma}
\]
\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} (-1)^{1+s\rho+s\sigma} \sin \left( \frac{\pi \rho}{3} \right) \cos \left( \frac{\pi \rho}{4} \right) \sin \left( \frac{12\pi s \sigma}{p} \right)
\]
\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{1s;\rho\sigma} \chi_{\rho\sigma}
\]

Note that \(A_{1s;1\sigma} = A_{1s;7\sigma}, A_{1s;4\sigma} = A_{1s;8\sigma}\) (since \(p\) is odd), \(A_{1s;5\sigma} = A_{1s;11\sigma}\), and all other \(A_{1s;\rho\sigma} = 0\), so

\[C_{1s}(-1/\tau) = 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{1s;\rho\sigma} C_{\rho\sigma}(\tau)\]

where \(\rho\) takes the values 1, 4, 5 which correspond to the blocks.

Similarly,

\[
C_{4s}(-1/\tau) = 2 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} (-1)^{1+s\rho} \sin \left( \frac{4\pi \rho}{12} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) + \sin \left( \frac{8\pi \rho}{12} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) \right) \chi_{\rho\sigma}
\]
\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} (-1)^{1+s\rho+s\sigma} \sin \left( \frac{\pi \rho}{2} \right) \cos \left( \frac{\pi \rho}{6} \right) \sin \left( \frac{12\pi s \sigma}{p} \right)
\]
\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{4s;\rho\sigma} \chi_{\rho\sigma}
\]

\(A_{4s;1\sigma} = A_{4s;7\sigma}, A_{4s;4\sigma} = A_{4s;8\sigma}, A_{4s;5\sigma} = A_{4s;11\sigma}\), and all other \(A_{4s;\rho\sigma} = 0\), so

\[C_{4s}(-1/\tau) = 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{4s;\rho\sigma} C_{\rho\sigma}(\tau)\]

where \(\rho\) takes the values 1, 4, 5 which correspond to the blocks.

Finally,

\[C_{5s}(-1/\tau) = 2 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} (-1)^{1+s\rho+s\sigma}.\]
\[
\left( \sin \left( \frac{5\pi \rho}{p} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) + \sin \left( \frac{11\pi \rho \sigma}{12} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) \right) \chi_{\rho \sigma}
\]

\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} (-1)^{1+s \rho + \sigma} \sin \left( \frac{2\pi \rho \sigma}{3} \right) \cos \left( \frac{\pi \rho}{4} \right) \sin \left( \frac{12\pi s \sigma}{p} \right)
\]

\[
= 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{5s;\rho \sigma} \chi_{\rho \sigma}
\]

\[A_{5s;1\sigma} = A_{5s;7\sigma}, \; A_{5s;4\sigma} = A_{5s;8\sigma}, \; A_{5s;5\sigma} = A_{5s;11\sigma}, \; \text{and all other} \; A_{5s;\rho \sigma} = 0, \; \text{so}
\]

\[C_{5,s}(-1/\tau) = 4 \sqrt{\frac{1}{6p}} \sum_{\rho,\sigma} A_{5s;\rho \sigma} C_{\rho \sigma}(\tau)
\]

where \( \rho \) takes the values 1, 4, 5 which correspond to the blocks.

To check unitarity, we take \( S \) in the block basis, with coefficients \( 4/\sqrt{6p} A_{rs;\rho \sigma} \) as above, and compute \( S^\dagger S \). Since \( S \) is real, we just have to show

\[A^\dagger A = \frac{3p}{8} I\]

where \( I \) is the \( 3(p - 1)/2 \times 3(p - 1)/2 \) identity matrix. For each \((r, s), \ (r', s)\), we compute

\[\sum_{\rho,\sigma} A_{rs;\rho \sigma} A_{r's';\rho \sigma}\]

For example, if \( r = r' = 1 \), this is

\[\sum_{\rho,\sigma} (-1)^{\rho(s + s')} \sin \left( \frac{\pi \rho \sigma}{3} \right) \cos \left( \frac{\pi \rho}{4} \right) \sin \left( \frac{12\pi s \sigma}{p} \right) \sin \left( \frac{12\pi s' \sigma}{p} \right)\]

We have that

\[\sum_{\sigma} \sin \left( \frac{12\pi s \sigma}{p} \right) \sin \left( \frac{12\pi s' \sigma}{p} \right) = \begin{cases} p/4 & s = s' \\ 0 & \text{otherwise} \end{cases}\]

Note that this part of the calculation is the same for any \( r, r' \). This already implies the vanishing of the off-diagonal terms of the product with \( s \neq s' \).
Then we put \( s = s' \) and are led to calculate

\[
\sum_{\rho} \sin \left( \frac{\pi p \rho}{3} \right)^2 \cos \left( \frac{\pi p \rho}{4} \right)^2 = \frac{3}{2}.
\]

Thus the 1s; 1s entry of the product matrix is

\[
\frac{3}{2} \cdot \frac{p}{4} = \frac{3p}{8}
\]

as desired. The cases of other \( r, r' \) are similar.


**Solution:** This is a simple substitution of the matrix elements of the \( S \)-transformation of the minimal model into the Verlinde formula. As the book suggests, the simplest way to do this is to write a simple computer program, although it can be done by hand as well. Using the nomenclature for the fields from Chapter 7 of the text, the nontrivial fusion rules are:

\[
M(5, 2)
\]

\[
\Phi \times \Phi = 1 + \Phi
\]

\[
M(4, 3)
\]

\[
\epsilon \times \epsilon = 1
\]

\[
\epsilon \times \sigma = \sigma
\]

\[
\sigma \times \sigma = 1 + \epsilon
\]
\[ M(5, 4) \]

\[
\begin{align*}
\epsilon \times \epsilon &= 1 + \epsilon' \\
\epsilon \times \epsilon' &= \epsilon + \epsilon'' \\
\epsilon \times \epsilon'' &= \epsilon' \\
\epsilon \times \sigma &= \sigma + \sigma' \\
\epsilon \times \sigma' &= \sigma \\
\epsilon' \times \epsilon' &= 1 + \epsilon' \\
\epsilon' \times \epsilon'' &= \epsilon \\
\epsilon' \times \sigma &= \sigma + \sigma' \\
\epsilon' \times \sigma' &= \sigma \\
\epsilon'' \times \epsilon'' &= 1 \\
\epsilon'' \times \sigma &= \sigma' \\
\epsilon'' \times \sigma' &= \sigma' \\
\sigma \times \sigma &= 1 + \epsilon + \epsilon' + \epsilon'' \\
\sigma \times \sigma' &= \epsilon + \epsilon' \\
\sigma' \times \sigma' &= 1 + \epsilon''
\end{align*}
\]