1. Verify formulas (7.62) and (7.65) in the text.

**Solution:** (7.62): Since the dotted line has equation $\alpha_+ x + \alpha_- y = 0$, the distance from the point $(r, s)$ to the line is

$$\delta = \frac{\alpha_+ r + \alpha_- s}{\sqrt{\alpha_+^2 + \alpha_-^2}}.$$ 

Then

$$h_0 + \frac{\delta^2}{4} (\alpha_+^2 + \alpha_-^2) = h_0 + \frac{1}{4} (\alpha_+ r + \alpha_- s)^2 = h_{r,s}.$$ 

(7.65) From $p\alpha_+ + p'\alpha_- = 0$ we get the equation

$$p \left( \frac{\sqrt{1-c} - \sqrt{25-c}}{\sqrt{24}} \right) + p' \left( \frac{\sqrt{1-c} + \sqrt{25-c}}{\sqrt{24}} \right)$$

which can be rewritten as

$$(p + p') \sqrt{1-c} = (p - p') \sqrt{25-c}.$$ 

Squaring both sides gives

$$\left( p^2 + 2pp' + (p')^2 \right) (1-c) = \left( p^2 - 2pp' + (p')^2 \right) (25-c)$$

$$-24p^2 + 52pp' - 24 (p')^2 = 4pp'c$$

Rewriting the lhs as $-24(p - p')^2 + 4pp'$ we get the desired result

$$c = 1 - 6\frac{(p - p')^2}{pp'}.$$ 

For the $h_{r,s}$ we note from the above that

$$h_0 = \frac{c - 1}{24} = -\frac{(p - p')^2}{4pp'} \tag{1}$$
Next, from
\[ \sqrt{1-c} = (p - p') \sqrt{\frac{6}{pp'}}, \quad \sqrt{25-c} = (p + p') \sqrt{\frac{6}{pp'}}, \]
we get
\[ \alpha_+ = \frac{p}{\sqrt{pp'}}, \quad \alpha_- = -\frac{p'}{\sqrt{pp'}} \]
so that
\[ \frac{1}{4} (r \alpha_+ + s \alpha_-)^2 = \frac{(rp - sp')^2}{4pp'} \]
(The 2) Combining (1) and (2) gives
\[ h_{r,s} = h_0 + \frac{1}{4} (r \alpha_+ + s \alpha_-)^2 = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \]

2. Consider the minimal model $\mathcal{M}(5, 3)$.

(a) Find the central charge, and list all the primaries and their weights.

**Solution:** From $(p, p') = (5, 3)$ we get $c = 1 - 24/15 = -3/5$. The weights are
\[ h_{r,s} = \frac{(5r - 3s)^2 - 4}{60} \]
This gives the primaries and weights
\[
\begin{array}{ccc}
\phi_{r,s} & h_{r,s} \\
\phi_{1,1} = \phi_{2,4} & 0 \\
\phi_{2,1} = \phi_{1,4} & \frac{3}{5} \\
\phi_{2,2} = \phi_{1,3} & \frac{1}{5} \\
\phi_{2,3} = \phi_{1,2} & -\frac{1}{20}
\end{array}
\]

(b) Write down all the fusion rules, truncated as much as possible. Since $\phi_{1,1}$ is the identity operator, truncated as much as possible. Since $\phi_{1,1}$ is the identity operator, truncated as much as possible.

Since $\phi_{1,1}$ is the identity operator, we have $\phi_{1,1} \times \phi_{r,s} = \phi_{r,s}$ for all $(r, s)$.
\[
\begin{align*}
\phi_{2,1} \times \phi_{2,1} &= \phi_{1,1} \\
\phi_{2,1} \times \phi_{2,2} &= \phi_{1,2} \quad = \phi_{2,3} \\
\phi_{2,1} \times \phi_{2,3} &= \phi_{1,3} \quad = \phi_{2,2} \\
\phi_{2,2} \times \phi_{2,2} &= \phi_{1,1} + \phi_{1,3} \quad = \phi_{1,1} + \phi_{2,2} \\
\phi_{2,2} \times \phi_{2,3} &= \phi_{1,2} + \phi_{1,4} \quad = \phi_{2,3} + \phi_{2,1} \\
\phi_{2,3} \times \phi_{2,3} &= \phi_{1,1} + \phi_{1,3} \quad = \phi_{1,1} + \phi_{2,2}
\end{align*}
\]
(c) Find the first six levels at which singular vectors appear in the Verma module \( V_{1,2} \).

**Solution:** We have the table of Verma modules

\[
\begin{array}{ccc}
V_{4,3} = V_{2,7} & \rightarrow & V_{7,2} = V_{2,13} & \rightarrow & V_{10,3} = V_{2,17} \\
\uparrow & & \downarrow & & \nearrow \\
2 & & 14 & & 28 \\
V_{1,2} = V_{2,3} & \rightarrow & V_{5,2} = V_{1,8} & \rightarrow & V_{8,3} = V_{1,12} & \rightarrow & V_{11,2} = 1,18 \\
\downarrow & & 6 & & 16 & & 40 \\
\end{array}
\]

With vanishing levels (relative to \( h_{1,2} \)) written below the top row and above the bottom row. Diagonal arrows are omitted. So the first six levels at which singular vectors occur are 2, 6, 14, 16, 28, 40.

(d) Find the character of the irreducible representation spanned by \(|h_{1,2}\rangle\), expanded up to order \( q^{h_{1,2} - c/24 + 8} \).

**Solution:** We get the character

\[
\chi_{1,2}(q) = \frac{q^{h_{1,2} - c/24}}{\phi(q)} \left(1 - q^2 - q^6 + O(q^9)\right)
\]

which expands to

\[
q^{-1/40} \left(1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 7q^7 + 9q^8 + O(q^9)\right)
\]