1. In this problem you are to identify certain holomorphic transformations in 2 Euclidean dimensions as special conformal transformations.

(a) Check that the transformations $T_a$ of $S^2 = \mathbb{C} \cup \{\infty\}$ defined by

$$T_a(z) = \frac{z}{1 + a z}$$

are closed under composition hence form a transformation group. Here $a$ is a complex number.

Solution:

$$T_a(T_b(z)) = T_a\left(\frac{z}{1 + bz}\right) = \frac{z/(1 + bz)}{1 + az/(1 + bz)} = \frac{z}{1 + (a + b)z} = T_{a+b}(z).$$

Alternatively, $T_a$ is identified with the SL(2, $\mathbb{C}$) transformation

$$T_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

and then is is clear that $T_a T_b = T_{a+b}$.

(b) Show that infinitesimal generators of this group are given by $L_1 + \bar{L}_1$ and $i(L_1 - \bar{L}_1)$.

Solution: $T_a(z) = z - az^2 + O(a^2)$ while $\overline{T_a(z)} = \bar{z} - \bar{a}\bar{z} + O(a^2)$. For $a$ real this gives the generator

$$-z^2 \partial_z - \bar{z}^2 \partial_{\bar{z}} = L_1 + \bar{L}_1$$

while for pure imaginary $a$ we get the generator

$$-iz^2 \partial_z + i\bar{z}^2 \partial_{\bar{z}} = i \left( L_1 - \bar{L}_1 \right).$$

(c) Identify $T_a$ as a special conformal transformation.

Solution: The general form of a SCT is

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}.$$
Writing

\[ T_a(z) = \frac{z}{1 + az} = \frac{z (1 + \bar{a} \bar{z})}{(1 + az) (1 + \bar{a} \bar{z})}, \]

considering the complex conjugation equation, and remembering that the metric is \( g_{z \bar{z}} = \frac{1}{2} \) so that \( z \cdot \bar{z} = |z|^2 \) etc, we recognize \( T_a \) as a SCT with \( b^z = -\bar{a} \) and \( \bar{b} \bar{z} = -a \).

2. Let \( \phi(z) \) be a free boson.

(a) Complete the calculation showing that \( \partial \phi(z) \) is primary of weights \((h, \bar{h}) = (1, 0)\).

**Solution:** We computed the \( T(z) \phi(w) \) OPE in class to verify that \( h = 0 \), so we only have to compute the \( \bar{T}(\bar{z}) \partial \phi(w) \) OPE

\[ \bar{T}(\bar{z}) \partial \phi(w) = -2\pi g : \bar{\partial} \phi(\bar{z}) \bar{\partial} \phi(\bar{z}) : \partial \phi(w) = -2\pi g (2) (\bar{\partial} \phi(\bar{z}) \partial \phi(w)) \bar{\partial} \phi(\bar{z}). \]

Now

\[ \langle \bar{\partial} \phi(\bar{z}) \partial \phi(w) \rangle = 0 \]

so the OPE has no singularities. On the other hand \( \bar{\partial} \partial \phi = 0 \) as an operator, so we do have with \( \bar{h} = 0 \)

\[ \bar{T}(\bar{z}) \partial \phi(w, \bar{w}) \sim \frac{\bar{h}}{(\bar{z} - \bar{w})^2} + \frac{\bar{\partial} \partial \phi(w, \bar{w})}{\bar{z} - \bar{w}} = 0 \]

and we have verified the required OPE.

(b) Show that for any \( k \)

\[ : e^{k \phi(z)} := 1 + k \phi(z) + \frac{k^2}{2} : \phi(z) \phi(z) : + \ldots \]

is primary, and find its weights.

**Solution:** We compute

\[ T(z) \frac{k^n}{n!} : \phi(w)^n := -\frac{2\pi g k^n}{n!} : \partial \phi(z) \partial \phi(z) :: \phi(w)^n : \]

We can contract twice or once, obtaining

\[ -\frac{(k^n/8\pi g) : \phi(w)^{n-2}/(n-2)! :}{(z - w)^2} + \frac{(k^n/(n-1)!) : \partial \phi(z) \phi(w)^{n-1} :}{(z - w)} \]
After Taylor expanding, the singular part of the second term can be rewritten as

\[ k^n \partial \left( \frac{\phi^n(w)}{n!} \right) : z - w : \]

Summing over \( n \), and considering essentially the same calculation for the OPE with \( \bar{T} \) we see that \( e^{ik\phi(z)} : \) is primary of weights \( -k^2/8\pi g, -k^2/8\pi g \).

Comment: it would have been more appropriate to discuss the operators \( e^{ik\phi(z)} : \) which have weights \( (k^2/8\pi g, k^2/8\pi g) \) and will be identified with vertex operators which create a “vacuum” state.