Earlier studies of CFT were done on flat 2d complex plane. Now we would like to study CFT on compact 2d surfaces.

There are two possible compact spaces in 2d:

a) Sphere
b) Torus

case (a) is the special case of flat plane as we've discussed earlier. There exist conformal transformations that maps a physical theory on a flat complex plane to a spherical 2d space.

case (b) is interesting because its application to string theory. String theory, as one may recall, is a 2d world sheet, and therefore scans a cylindrical space-time as it progresses in time.

Fields are defined on \((C, t)\) and therefore the relevance of compact 2d surfaces become important.

Therefore it seems cylinder is logically possible, so the question is when would a torus be useful?

- This is where "loop" diagrams or string theory gone nice to torus (or more generic tori)
the former is a Feynman diagram for fields and the latter is the corresponding string diagram.

Thus CFT on a torus is relevant, and the immediate next question would be to study:

a) CFT of bosons on a torus of genus $g$.
b) CFT of fermions on a torus of genus $g$.

One way of studying the CFT on a given space is via the partition function $Z$ defined as

$$Z = \text{tr} \, e^{-L_0 - c/24} = \text{tr} \, e^{-L_0 - c/24}$$

where $L_0$, $L_0$ are the Virasoro generators and $c$ is the central charge. Various relevant things are:

a) modular transformations
b) Lehman twistsc) fundamental domains
d) orbifolding

Some of which have been discussed in earlier lectures.

There is also a simple way to see that genus $g = 0$ is a sphere: Start with $g = 1$ torus

and shrink one of the cycles to zero size

and shrink one of the cycles to zero size
parallel discussion of spin-structures

Two non-contractible loops on a torus of genus $g=1$

($++$)
($+-$) four boundary condition
($-+$)
($--$)

- A spin structure is even if the number of zero modes of chiral Dirac operator is even. (and is odd otherwise)

- On a flat torus chiral Dirac operator is $\mathbb{Z}$ ($++$) bound condition allows for a constant spinor, and this is the zero mode of $\mathbb{Z}$.

Define complex coordinates on a torus
\[ z = z_1 + \tau z_2, \quad \bar{z} = \bar{z}_1 + \tau \bar{z}_2 \]
\[ \tau = \text{complex structure}. \]

\[ \gamma (z^1 + 1, z^2) = \pm \gamma (z^1, z^2) \]
\[ \gamma (z^1, z^2 + 1) = \pm \gamma (z^1, z^2) \]

This leads to four spin structures.
One another question to ask while studying CFT on compact space is distribution of modes of the theory. For a generic Quantum field theory, we know that there're INFINITE number of harmonic oscillators. When the space becomes compact, these modes have integer momenta, and therefore a countable infinite number of harmonic oscillators.

Each of these modes are individually a quantum mechanical system with its corresponding quantum wavefunction. Together they define a quantum field theory on a compact space (time is still infinite!)

\[
\Psi = Tr \ e^{-\frac{1}{\hbar} \int_{\Sigma} \left( H \frac{\partial}{\partial z} - i P \frac{\partial}{\partial \bar{z}} \right) dz d\bar{z}}
\]

\[
\equiv \sum_{n} \langle n | e^{-\frac{1}{\hbar} \int_{\Sigma} \left( H \frac{\partial}{\partial z} - i P \frac{\partial}{\partial \bar{z}} \right) dz d\bar{z}} | n \rangle
\]

\[H \equiv \text{Hamiltonian} = L_0 + \bar{L}_0 - \frac{c}{12}\]

\[P \equiv \text{momentum} = \frac{2\pi L}{\hbar} (L_0 - \bar{L}_0)\]

and \( | n \rangle \) is number state with harmonic-oscillator wavefunction.

- Partition function for bosons on a genus 0 surface
- Partition function for fermions on a genus 0 surface
**Partition function for bosons on a torus**

\[ Z_{\text{bos}}(x) = \left( \text{Im} \, \varphi \right)^{-\frac{1}{2}} | \eta(x) |^{-2} \]

which is modular invariant. The above calculation can also be done by actually evaluating the path integral. This has been done explicitly in the previous lectures.

We’re also ignoring the zero mode.

Q: Why are the zero modes problematic?
A: The zero modes are no longer discrete, and therefore do not have harmonic oscillator wave functions. In fact, their wave functions are like free particle ones, and they’re continuous.

In terms of string world sheet, they’re related to free motion of the string in space-time.

Thus for free bosons we’d take the field \( \Phi \) to have the following mode expansions

\[ \Phi(x) = \sum_n \psi_n(x) \]

Observe that the sum is over integral \( n \), and therefore reflects the finiteness of space.

Similarly one can do the partition function of fermions on a torus

\[ d\alpha = 0, \quad dv_x = \sqrt{\frac{\theta_3}{\eta}}, \quad dv_y = \sqrt{\frac{\theta_4}{\eta}}, \quad dw = \sqrt{\frac{\theta_2}{\eta}} \]

\[ Z_{\text{ferm}} = |dw|^{-2} \]

\[ Z = |\frac{\theta_2}{\eta}| + \frac{\theta_3}{\eta} + \frac{\theta_4}{\eta} \]

is the modular invariant partition function on a torus.
We can also study CFT in terms of free fields. In general quantum field theory the free expectation value can be constant, or can be a time independent configuration. The latter is called solitonic solution. In the presence of a solitonic field, the field \( \phi \) (bosonic field \( \phi \)) is decomposed in the following way:

\[
\phi = \phi_{\text{free}} + \delta \phi
\]

where \( \delta \phi \) = fluctuation, and \( \phi_{\text{free}} \) = bos. solitonic field.

For the CFT we could also study such configuration in terms of free field as \( \phi_{\text{free}} \).

Following Francesco et. al we write this as

\[
\phi = \phi^{(t)}_{m,m'} + \phi
\]

The partition function becomes

\[
Z_{m,m'} = Z_{\text{free}} (t) e^{-\frac{\pi R^2 |m^2 - m'^2|}{2 i m t}}
\]

The full partition function becomes

\[
Z = R^{\frac{D}{2}} Z_{\text{free}} (t) \sum_{m,m'} e^{-\frac{\pi R^2 |m^2 - m'^2|}{2 i m t}}
\]

\[
= \frac{1}{|m|^2} \sum_{e_m \in Z} q^{\frac{1}{2} \left( \frac{e_m}{2} + \frac{m^2}{2} \right)^2 - \frac{1}{2} \left( \frac{e_m}{2} - \frac{m^2}{2} \right)^2} q^2
\]

This basically implies the following remarkable relation:

\[
Z \left( \frac{2 \pi i}{R} \right) = Z (R)
\]

i.e. theory at Radini \( R \) is \textit{same} as theory at radini \( \frac{2 \pi}{R} \).

In string theory, this is called \textit{T-duality}.
what happen when we have more than one scalar fields?
the vertex generator can be generalised to

\[ L_0 = \frac{p^2}{2} + \frac{\alpha}{\beta} \sum_{k \geq 0} \frac{\alpha_k}{k+1} \frac{1}{2} \]

and similarly \( \tilde{L}_0 \). Here we've summed over \( n \) bosonic fields.
the partition function will be

\[ Z(\tau) = \frac{1}{\eta^{\frac{1}{24}}} \sum_{\gamma \in \Gamma} e^{i \pi \tau \gamma^2 - i \pi \gamma \bar{\gamma}} \]

and \( \Gamma \), \( \bar{\gamma} \) a \( \mathbb{R} \times \mathbb{R} \) lattice. Modular transformation gives

\[ Z(\tau + 1) = Z(\tau) e^{\frac{2\pi^2 (\eta - \bar{\eta})}{24}}. \]

Alternatively:

in terms of field components, this is:

\[ \phi^I = \phi^I + \sqrt{2} \pi \sum_{\alpha=1}^{D} \bar{R}_\alpha \cdot e^I \bar{e}_\alpha = \phi^I + 2\pi l^I \]

with \( l^I = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^{D} R_\alpha \cdot e^I \bar{e}_\alpha \)

the \( \bar{e}_\alpha = \sum_{\alpha=1}^{D} e^I \bar{e}_\alpha \) \( (i = 1, \ldots, D) \) are \( D \) linearly independent vectors normalized to \( (\bar{e}_\alpha)^2 = 2 \)

the \( \bar{L} = \sum_{\alpha=1}^{D} L_\alpha \) can be thought of a lattice vectors of a \( D \) dimensional lattice \( \Lambda \): \( \bar{L} \in \Lambda \)

therefore the torus on which we compactify is obtained by dividing \( \mathbb{R}^D \) by \( 2\pi \Lambda \)

\[ T^D = \mathbb{R}^D / 2\pi \Lambda \]
we'd also have

\[ [q^i, p^j] = i \delta^{ij} \]

This is the GM relation, and implies that \( p^i \)
generates translation \( \epsilon \) \( q^i \) \( \epsilon \) and single valuedness
\( e^{i} \alpha^{i} p^{i} \) requires \( L^2 p^i \in \mathbb{Z} \), i.e. the allowed
momenta have to lie on the lattice which is dual to \( \Lambda^D \), denoted by \( (\Lambda^D)^* \):

\[ p^i = \sqrt{2} \sum_{n=1}^{D} \frac{m_n}{R_i} e^n_i \]

where \( e^n_i \) is dual to \( e^n_i \), i.e.

\[ \sum_{n=1}^{D} e^n_i e^n_j = \delta_{ij} \]

and the basis vectors of \( (\Lambda^D)^* \) are \( \frac{\sqrt{2}}{R_i} e_i^* \)

The metric on the lattice is

\[ g_{ij} = \frac{1}{2} \sum_{I=1}^{D} R_i e^I_i R_j e^I_j \]

\[ g^{* \ast}_{ij} = 2 \sum_{I=1}^{D} \frac{1}{R_i} e^{* \ast I}_i R_j e^I_j \]

\[ \therefore \quad g^{* \ast}_{ij} = (g^*)_{ij} \]

\[ \circ \quad \text{vol} (\Lambda^D) = \sqrt{\text{det} \, g} \quad \text{vol} (\Lambda^D)^* = \frac{1}{\sqrt{\text{det} \, g}} \]

As a simple illustration, consider the case for a one dimensional lattice.

a) In CFT there are massless vector states given by the internal oscillations of the vacuum.

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(b) There are massive vector states that arise from solitons.

- Together they form the $SU(2)_L \times SU(2)_R \rightarrow U(1)_L \times U(1)_R$ realization in this CFT.
- At the self-dual radius, full $SU(2)_L \times SU(2)_R$ symmetry is restored.
  Thus $U(1)_L \times U(1)_R$ : Cartan subalgebra
  and the solitonic states are the roots!

- Enhancement of gauge symmetry in CFT, this is not possible for pt. particle case.

- For the $d$-dimensional case, Cartan will be $U(1)^D_L \times U(1)^D_R$

  - The general partition function is given by

$$Z = \text{Tr} q^{H_R} \bar{q}^{H_L}$$

  where

$$H_L = \frac{P_L^2}{2} + N_L + \frac{P_L^2}{2} - \bar{q}$$

$$H_R = \frac{P_R^2}{2} + N_R + \bar{q}$$

  where $\bar{q}$ used a very generic case in which left moving sector is compacted on a lattice and the right moving sector is not.

  Also right moving sector has fermions.

$$Z \sim \frac{1}{(\text{Int})^4} \left| \frac{1}{\eta} \right|^{24} \left( \frac{\sqrt{2}}{p_L} \right)^{\frac{1}{2} P_L^2} \frac{\eta_1^4 - \eta_4^4 - \eta_3^4}{\eta_2^{12}}$$
This is the partition function for a heterotic string.

Define \( P(z) = \sum_{\mathbf{P}_L} \mathbb{Z} \mathbf{P}_L^2 \)

- \( P(z+1) = P(z) \)
- \( P(-\frac{1}{2}) = e^{i\pi} P(z) \)

This is possible if \( \mathbf{P}_L \) forms a self-dual lattice constraint coming from modular invariance.

A similar story could be done for orbifolds.

\( \mathbb{Z}_2 \) orbit of a torus:

The shaded region is the orbifold.

\[
Z_{\text{orb}}(R) = \frac{1}{2} \left( Z(R) + \frac{|\theta_2 \theta_3|}{|\eta|^2} + \frac{|\theta_2 \theta_4|}{|\eta|^2} + \frac{|\theta_3 \theta_4|}{|\eta|^2} \right)
\]