1.2 Categories and Functors

Categories: objects and morphisms

- Collection of objects \( \text{obj}(C) \), \( \text{Ob}(C) \), or just \( C \)

- To any objects \( A, B \in C \), a set of morphisms \( \text{Mor}(A, B) \)

- Morphisms compose by a map of set \( \text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C) \)

Write \( f \in \text{Mor}(A, B) \) as \( f : A \rightarrow B \)

Composition as \( f \circ g : A \rightarrow C \)

- \( \forall A \in C \) \( \exists \text{Id}_A \in \text{Mor}(A, A) \) such that \( \forall B \in C, \forall f \in \text{Mor}(A, B) \) have \( f = f \circ \text{Id}_A \).
A morphism $f \in \text{Mor}(A, B)$ is an isomorphism if there exists a two-sided inverse $g \in \text{Mor}(B, A)$ such that $g \circ f = \text{id}_B$ and $f \circ g = \text{id}_A$.

Examples of categories:

- Sets and maps of sets $\text{Set}$
- Abelian groups and group homomorphisms $\text{Ab}$
- Modules over a ring $A$ and homomorphisms of modules $\text{Mod}_A$
- Topological spaces and continuous maps $\text{Top}$
- Differentiable manifolds and differentiable maps $\text{Diff}$
- Algebraic varieties and morphism of varieties $\text{Var}$
- Sheaves of abelian groups on a topological space $\text{Ab}(X)$
Functors

Categories $\mathcal{C}, \mathcal{D}$

$F : \mathcal{C} \rightarrow \mathcal{D}$ (covariant by default)

- $F : \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$
- $\forall C_1, C_2 \in \mathcal{C} \quad \text{Mor}_\mathcal{C}(C_1, C_2) \rightarrow \text{Mor}_\mathcal{D}(F(C_1), F(C_2))$

preserving composition and identities

Examples

- Forgetful functors, e.g. $F : \text{Ab} \rightarrow \text{Sets}$, $F(G) = G$
  ("forget" that $G$ is a group)

- Representable functors $\forall A \in \mathcal{C}$ have
  $h^A : \mathcal{C} \rightarrow \text{Sets}$ $h^A(B) = \text{Mor}(A, B)$
  $h^A : \text{Mor}(B, C) \rightarrow \text{Mor}(h^A(B), h^A(C))$, $h^A(f) : \{ f : B \rightarrow C \}$
Terminology

\[ F : C \to D \text{ is faithful if } \forall C, C' \]
\[ \text{Mor}_A(C, C') \to \text{Mor}_B(F(C), F(C')) \text{ is inj.} \]

\[ \text{Full if } \forall C \text{ such that } \text{Mor}_A(C, C') \neq \emptyset \]

\[ \text{Fully faithful } \Leftrightarrow \text{isomorphism} \]

\[ \mathcal{C} \subseteq \mathcal{D} \text{ full subcategory : } i : \mathcal{C} \to \mathcal{D} \]
Contravariant functor: reverse the arrows!

\[ A \rightarrow B \rightarrow C \]

\[ F(A) \leftarrow F(B) \leftarrow F(C) \]

Example:

\[ X \mapsto H^1(X, \mathbb{Z}) \] contravariant Tor to Ab

Representable functor (functor of points):

\[ A \in C \quad h_A : C \rightarrow \text{Set} \] contravariant

\[ h_A(C) = \text{Mor}(C, A) \]

\[ B \rightarrow C \quad h_A(C) \rightarrow h_A(B) / \]

\[ \{ C \rightarrow A \} \rightarrow [C \rightarrow A] \rightarrow C \xrightarrow{f} A \]
Opposite category $\mathcal{C}^{\text{op}}$ to $\mathcal{C}$

reverses the arrows

Then a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$

is identified with a covariant

$F : \mathcal{C}^{\text{op}} \to \mathcal{D}$
Natural Transform between two functors

Given categories $C$, $D$.

Functors $F, G : C \rightarrow D$.

A natural transformation $F \Rightarrow G$ is:

- For each $C \in C$, a morphism $m_C : F(C) \rightarrow G(C)$ s.t.
  
  $F(C) \xrightarrow{m_C} G(C)$
  
  $F(c) \downarrow C \Rightarrow G(C)$

  $F(c) \xrightarrow{m_{c'}} G(C')$

Natural isomorphism: Each $m_C$ is isomorphism.
This is an equivalence relation on categories.
Equivalent categories may not be isomorphic.

Ex. (EU Ex 1.2.0) \( \text{Ob}(U) = \{ \mathbf{k}^n : n \in \mathbb{N} \} \)

\( \text{Ob}(\text{Vect}) \) finite dimensional vector spaces.

\( U \) and \( \text{Vect} \) are equivalent but not isomorphic.
Useful concepts defined by universal properties.

- Localization A ring, $S \subseteq A$ mult set
  
  Coarsest $A$-algebras $A \rightarrow B$ s.t. $\varphi(s)$ unit $\forall s \in S$
  
  $A \rightarrow S^{-1}A$ is universal for this property:

\[
\begin{array}{c}
A \\
\downarrow \varphi
\end{array}
\rightarrow
\begin{array}{c}
S^{-1}A \\
\downarrow E(4)
\end{array}
\rightarrow
\begin{array}{c}
B
\end{array}
\]

\[
\frac{a}{s} \in S^{-1}A;
\]

\[
\varphi \left( \frac{a}{s} \right) = \varphi(s)^{-1} \varphi(a)
\]
Tensor products of $A$-modules:

$M \otimes_A N$ universal for $A$-linear maps $P \in \text{Mod}_A$

$\begin{align*}
\forall M \times N & \rightarrow P \\
\Rightarrow & M \rightarrow M \otimes_A N
\end{align*}$

Fiber products (C arbitrary):

$\begin{align*}
\text{universal for } & M \rightarrow A \\
\text{universal for } & B \rightarrow C
\end{align*}$
Yoneda's Lemma: \( A, A' \in \text{Ob}(C); h_A, h_{A'} : C \to \text{Set} \)

The natural transformations \( h_A \to h_{A'} \) are in bijection with \( \text{Mor}_C(A, A') \)

**Pf (sketch):** Given \( \alpha : h_A \to h_{A'} \), apply to \( A \)

- Given \( f : A \to A' \)
- \( \alpha f : h_A \to h_{A'} \)
- \( \alpha f (a) : h_A(a) \to h_{A'}(a) \)
- \( \alpha f (a) : B \to A' \)
A non-empty category $\mathcal{D}$ is filtered if

\[ \forall x, y \in \mathcal{D}, \exists z \in \mathcal{D} \text{ such that } x \xrightarrow{f} z \xrightarrow{g} y. \]

Exercise: Suppose $\mathcal{D}$ is filtered, $F : \mathcal{D} \to \text{Sets}$, diagram $F(i) = S_i$.

\[ \text{Compare to explicit formula for stalks of sheaves.} \]
Adjoint functors

\[ F : \mathcal{C} \to \mathcal{D} \text{ left adjoint to } G : \mathcal{D} \to \mathcal{C} \]

\[ \text{leg} : \mathcal{C} \text{ right adjoint to } F \]

\[ \forall \mathcal{C} \in \mathcal{A}, \mathcal{D} \in \mathcal{B} \text{ have natural bijection} \]

\[ \mathsf{Mor}_\mathcal{B}(F(\mathcal{C}), \mathcal{D}) \cong \mathsf{Mor}_\mathcal{A}(\mathcal{C}, G(\mathcal{D})) \]
Ex.

1. X top space. C presheaves of abelian gp on X

\[ D = \text{Ab}(X) \]

\[ F : C \rightarrow D \]

\[ G : D \rightarrow C \text{ forgetful} \]

\[ \mathcal{F} \text{ presheaf} \quad (\mathcal{F} \in \mathcal{C}) \]

\[ G \text{ sheaf} \quad (g \in \text{Ab}(X)) \]

\[ \text{Mor}_{\text{Ab}(X)} (\mathcal{F}, \mathcal{G}) \leftrightarrow \text{Mor}_{\mathcal{C}} (\mathcal{F}, g) \]

\[ \mathcal{F} \rightarrow 
\]

Is just the statement of the universal property of the associated sheaf constructor.
Given \( \alpha : f \to fg \), construct

\[
\forall x, y \\
\alpha(x) = \lim_{u \to \alpha(x)} (u \circ f(x))
\]

By colimit jet, \( f \mapsto (\alpha(x)) \)
Abelian Categories

Not C is an additive category if

• ∀ A, B ∈ C, \( \text{Mor}_C(A, B) \) abelian gp

• ∀ A, B, C ∈ C group homomorphism

\[ \text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C) \]

C has a 0 object \( \text{(initial + final)} \)

C has products

\[
\begin{array}{ccc}
A_1 \times A_2 & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\quad \text{univ. for} \quad \begin{array}{ccc}
B & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\]

\[ \text{Limit} \quad \lim_{\theta} \]

\[ F : D ^{\circ} \rightarrow C \]

\[ F(1) = A_1, \quad F_{\theta}L \triangleq A_0 \]
In an additive category $\mathcal{C}$

$$\text{Hom}_\mathcal{C}(A, B) := \text{Mor}_\mathcal{C}(A, B)$$

morphisms in $\mathcal{C}$ called homomorphisms

$\mathcal{C}, \mathcal{D}$ additive categories

Define an additive functor $F : \mathcal{C} \to \mathcal{D}$

such that

- $F$ functor
- $\forall A, B \in \mathcal{C},$
  $$F : \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{D}(F(A), F(B))$$

honors structure of groups
In a category with 0 object

A kernel of $f: A \rightarrow B$ is a morphism $i: C \rightarrow A$ s.t.

- $f \circ i = 0$
- $i$ universal for this property

We already defined co-kernels, a colimit.

A kernel is a limit of diagram

$$
\begin{array}{c}
A \\
\downarrow \\
0 \\
\rightarrow B
\end{array}
$$
In any category, a monomorphism $f : A \rightarrow B$ is defined if:

$$\forall C, g_1, g_2 : C \rightarrow A \text{ s.t. } f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

In this case, $A$ is a subobject of $B$.

Dual notion of epimorphism: an epimorphism $g : A \rightarrow B$ means $B$ is a quotient object of $A$. 

($A$ epimorphism, say $B$ is a quotient object of $A$)
**Def.** An abelian category is an additive category s.t.

- Every homomorphism has a kernel and a cokernel.
- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of its kernel.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow \psi \\
\text{Ker} \phi & \cong & \text{Coker} \psi
\end{array}
\]
Example

- $\text{Ab}$
- $\text{Mod}_A$
- $\text{Ab}(X)$

Let $f: A \to B$ be a morphism in a category $\mathcal{C}$.

**Def** $\operatorname{Im} f := \ker(\operatorname{coker} f)$

**Exercise** If $\operatorname{Im} f$ exists, $f$ factors as

$A \xrightarrow{\alpha} \operatorname{Im} f \to B$,

a morphism

If $G$ is abelian

$\operatorname{Im} f = \operatorname{coker}(\ker f)$
Freyd - Mitchell Embedding Theorem

\[ \mathcal{C} \] is an abelian category. The \( \mathbb{Z} \)-ring \( A \)

and exact (takes exact sequences to exact sequences)

fully faithful

\[ \mathcal{C} \to \text{Mod}_A \]

embed \( \mathcal{C} \) as a full subcategory.

abstract

So many theorems about general abelian categories can be proven by diagram chasing, as if they were categories of modules!!