Math 514 Complex Algebraic Geometry

HOMEWORK 6 The case of Kahler Manifolds

Questions

1. Let $H_{\mathbb{R}}$ be a $\mathbb{R}$-vector space, and $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$

   (a) Show that a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$, where $H^{p,q} := H^{q,p}$ determines a continuous action $\rho : \mathbb{C}^* \to GL(H_{\mathbb{C}})$ of $\mathbb{C}^*$ on $H_{\mathbb{C}}$, given by $z \cdot \alpha^{p,q} = z^{p} \bar{z}^{q} \alpha^{p,q}$.

   Show that this action satisfies $\rho(z) = z \cdot \alpha^{p,q}$ where the conjugacy on $GL(H_{\mathbb{C}})$ is defined by $\bar{g}(u) = g(\bar{u})$.

   Show that one also has $\rho(t) = t^k \cdot Id$ for $t \in \mathbb{R}^*$.

   To show that this actually defines an action, note that the compatibility is proven by $z \cdot (w \cdot \alpha^{p,q}) = z^{p} \bar{z}^{q}(w^{p} \bar{w}^{q} \alpha^{p,q}) = z^{p} w^{p} \bar{z}^{q} \bar{w}^{q} \alpha^{p,q} = (zw) \cdot \alpha^{p,q}$.

   Also 1 acts as the identity: $1 \cdot \alpha^{p,q} = \alpha^{p,q}$.

   We note that by assumption $\alpha^{p,q} \in H^{q,p}$, so we get $\rho(z) \alpha^{p,q} = \rho(z) \alpha^{p,q} = z^{p} \bar{z}^{q} \alpha^{p,q} = \rho(z) \alpha^{p,q}$.

   So that $\rho(z) = \rho(z)$.

   For a nonzero real number $t$, $\rho(t) \alpha^{p,q} = t^{p} t^{q} \alpha^{p,q} = t^k \alpha^{p,q}$ where $p+q = k$. Thus $\rho(t) = t^k \cdot Id$.

   (b) Conversely, let $\rho : \mathbb{C}^* \to GL(H_{\mathbb{C}})$ be an algebraic action of $\mathbb{C}^*$ on $H_{\mathbb{C}}$ satisfying $\rho(t) = t^k \cdot Id$ for $t \in \mathbb{R}^*$ and $\rho(z) = \rho(z)$ for $z \in \mathbb{C}^*$. Applying the diagonalisation theorem for the actions of torsion abelian groups to the torsion points of $\mathbb{C}^*$, show that there’s a decomposition into direct sum

   $H = \bigoplus_{\chi} H_{\chi}$

   where $\chi$ is a character of $\mathbb{C}^*$, where $\mathbb{C}^*$ acts by $z \mapsto \chi(z) \cdot Id$ on $H_{\chi}$.

   The torsion subgroup of $\mathbb{C}^*$ is

   $T = \{ \zeta | \exists n \in \mathbb{Z} \text{ such that } \zeta^n = 1 \}$.

   Since $\mathbb{C}^*$ is commutative, $\rho(z)$ and $\rho(w)$ commute with each other for any $z,w \in \mathbb{C}^*$. Recall that a matrix is diagonalizable when its minimal polynomial has distinct roots (i.e. all roots have multiplicity 1). Since $\zeta^n = 1$, the minimal polynomial of $\rho(\zeta)$ divides $x^n - 1$, and thus has distinct roots. As a result, the $\rho(\zeta)$ are all diagonalizable.

   By the diagonalisation theorem, the action of $\rho(\zeta)$ for all $\zeta \in T$ is diagonalizable, and since they all commute they are simultaneously diagonalizable, and we get a basis of simultaneous eigenspaces. We can associate to each simultaneous eigenspace a function $\chi : T \to \mathbb{C}^*$ by sending $\zeta \in T$ to the eigenvalue, that is

   $\rho(\zeta)(v) = \chi(\zeta)v, \quad v \in H_{\chi}$.
where we have indexed the eigenspace with the same \( \chi \), denoting it as \( H_\chi \). It is immediate to check that \( \chi \) is a character of \( T \): letting \( \zeta, \zeta' \in T \), we compute by applying \( \rho(\zeta \zeta') = \rho(\zeta) \circ \rho(\zeta') \) to \( v \in H_\chi \) that
\[
\rho(\zeta \zeta')(v) = \chi(\zeta \zeta')v = \rho(\zeta) \circ \rho(\zeta')(v) = \chi(\zeta) \chi(\zeta')v,
\]
so that \( \chi(\zeta \zeta') = \chi(\zeta) \chi(\zeta') \) as required. We now have to extend the character \( \chi \) from \( T \) to \( \mathbb{C}^* \).

For all \( \zeta \in T \) we have an action \( \chi(\zeta) \) on \( H_\chi \). Moreover, since the action is continuous and \( T \) is dense in the unit circle, \( \chi \) extends to the unit circle \( \{|z| = 1\} \). On the other hand we also know that \( \rho(t) = t^k \text{Id} \) for \( t \in \mathbb{R} \). Since we can write any complex number uniquely as a product of a real number and an element of the unit circle
\[
z = \left( \frac{z}{|z|} \right) |z|,
\]
we can extend the character to \( \mathbb{C}^* \) by combining the actions of the reals and the unit circle:
\[
\chi(z) = \chi \left( \frac{z}{|z|} \right) |z|^k.
\]
Extending the character from \( T \) to \( \mathbb{C}^* \) continues to preserve the eigenspaces since we have only modified that action by real scalar multiplications, which trivially preserve all eigenspaces. Hence we have a decomposition
\[
H_\mathbb{C} = \bigoplus_\chi H_\chi.
\]

(c) Show that only the characters \( \chi_{p,q} : z \mapsto z^p \bar{z}^q \) with \( p + q = k \) appear in this decomposition.

Let \( \chi \) be the map \( T \to \mathbb{C}^* \). If \( \zeta^n = 1 \), then \( \chi(\zeta)^n = 1 \) as well, so \( \chi : T \to T \). Also, since \( \rho \) is continuous and \( T \) is dense in the unit circle, \( \chi \) extends to a continuous map \( S^1 \to S^1 \). Consider \( \chi^{-1}(1) \).

If \( \chi^{-1}(1) \) is infinite, then it’s dense in the unit circle. By the continuity of \( \chi \), it must be the trivial map \( \chi(\zeta) = \text{Id} \) for \( \zeta \in S^1 \). It follows that \( \chi(z) = |z|^k \) for \( z \in \mathbb{C}^* \). Here is the only place where we use the assumption that \( \rho \) is algebraic: since \( |z|^k = z^{k/2} \bar{z}^{k/2} \), we have that \( k \) must be even and \( p + q = (k/2) + (k/2) = k \).

If \( \chi^{-1}(1) \) is finite, then it consists of the \( n \)-th roots of unity for some \( n \), and so \( \chi(\zeta) = \zeta^l \text{Id} \) for \( l = n \) or \( l = -n \). Also, since \( \chi(-1) = (-1)^k \text{Id} \), we know that \( n \equiv k \mod 2 \).

Thus taking into consideration that \( \chi(t) = t^k \text{Id} \) for \( t \) real, we see
\[
\chi(z) = \chi(|z|) \chi(z/|z|) = |z|^k (z/|z|)^n \text{Id} = z^{\frac{k+n}{2}} \bar{z}^{\frac{k-n}{2}} \text{Id} = z^p \bar{z}^q \text{Id}
\]
Here we let \( p = \frac{k+n}{2} \) and \( q = \frac{k-n}{2} \), which are integers since \( n \equiv k \mod 2 \).

(d) Let \( H^{p,q} = H_{\chi_{p,q}} \). Show that \( H^{p,q} = \overline{H^{q,p}} \).

Let \( v \in H^{q,p} \). Then
\[
\rho(z) \cdot \bar{v} = \overline{\rho(z)} \cdot \bar{v} = \overline{\rho(z)} \cdot v = \bar{z}^p \bar{z}^q v = z^p \bar{z}^q \bar{v}
\]
Thus \( \bar{v} \in H_{\chi_{p,q}} \). Since the conjugation map is an involution, we can apply the argument to \( H^{p,q} \) instead of \( H^{q,p} \) and conclude that \( H^{q,p} = \overline{H^{p,q}} \), as desired.

2. The Hodge decomposition for curves. Let \( X \) be a compact connected complex curve.

(a) Show that \( d \) is surjective with kernel equal to the constant sheaf \( \mathcal{C} \). Hence we have an exact sequence
\[
0 \to \mathcal{C} \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \to 0
\]
To show surjectivity of $d$, we just need to show surjectivity on stalks. Let $z_0 \in X$ and $\omega \in \Omega^1_{X,z_0}$. Then we can find a neighborhood $U$ of $z_0$ on which $\omega$ can be represented by a holomorphic 1-form $\omega$, and, shrinking $U$ if necessary, we may assume that $U$ is simply connected and supports a chart with coordinate $z$. Then we can write $\omega = f dz$ with $f \in \mathcal{O}(U)$. Let

$$g(z) = \int_\gamma \omega$$

for any path $\gamma$ from $z_0$ to $z$ contained in $U$. Since $\omega$ is a holomorphic differential and $U$ is simply connected, this integral does not depend on the choice of the path $\gamma$. It follows that $g \in \mathcal{O}(U)$. By the fundamental theorem of calculus, we have $\omega = dg$. Thus $d$ is surjective.

To compute the kernel of $d$, note that for any open set $U$ and any $f \in \mathcal{O}(U)$, we have $df = 0$ if and only if $\frac{\partial f}{\partial z} = 0$ if and only if $f$ is locally constant. So the kernel of $d$ is the constant sheaf $\mathbb{C}$.

(b) Deduce from Serre duality that $H^1(X, \Omega_X) = \mathbb{C}$, and from Poincare duality that $H^2(X, \mathbb{C}) = \mathbb{C}$. By Serre duality $H^1(X, \Omega_X) \cong H^0(X, \mathcal{O}_X)^*$. But $H^0(X, \mathcal{O}_X) = \mathbb{C}$ since every holomorphic function on compact Riemann surface is a constant. Finally $\mathbb{C}^* \cong \mathbb{C}$.

By Poincaré duality $H^2(X, \mathbb{C}) \cong H_0(X, \mathbb{C})$, and since our Riemann surface is connected, $H_0(X, \mathbb{C}) \cong \mathbb{C}$.

(c) Show that the long exact sequence induced is

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

The long exact sequence associated to the above short exact sequence contains

$$H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \Omega_X) \to H^2(X, \mathbb{C}).$$

We just need to show that $H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X)$ is surjective and $H^1(X, \Omega_X) \to H^2(X, \mathbb{C})$ are injective.

Since $H^0(X, \mathcal{O}_X)$ is just the constant function, and every constant function in $H^0(X, \mathbb{C})$ is mapped to the same constant function in $H^0(X, \mathcal{O}_X)$, the first map is actually an isomorphism.

Since both $H^1(X, \Omega_X)$ and $H^2(X, \mathbb{C})$ have dimension 1, to prove that $H^1(X, \Omega_X) \to H^2(X, \mathbb{C})$ is injective, it suffices to show that this map is nonzero.

In fact, for the Kahler form $\omega$, we can think of it as an element of $H^1(X, \Omega_X)$ since it’s $\bar{\partial}$-closed by dimensional reason and is not $\bar{\partial}$-exact, as $\int_X \omega \neq 0$. On the other hand, it can be thought of as a representative of the second cohomology of the de Rham complex $C^\infty(X) \to A^1(X) \to A^2(X) \to 0$, since $\omega$ is $d$-closed by dimensional reason, and not $d$-exact, again since $\int_X \omega \neq 0$.

As a result, $[\omega]$ is mapped to $[\omega]$ and $H^1(X, \Omega_X) \to H^2(X, \mathbb{C})$ is an isomorphism as well.

(d) Show that the map which to a holomorphic form $\alpha$ associates the class $\tilde{\alpha} \in H^1(X, \mathcal{O}_X)$ is injective.

Since $\partial \alpha = 0$ for any holomorphic form $\alpha$, it follows that $\bar{\partial} \tilde{\alpha} = 0$, so that $\tilde{\alpha}$ represents a class $[\tilde{\alpha}] \in H^1(X, \mathcal{O}_X)$ by the Dolbeault isomorphism. We have to show that if $[\tilde{\alpha}] = 0$, then $\alpha = 0$.

To say that $[\tilde{\alpha}] = 0$ means that $\tilde{\alpha} = \bar{\partial} f$ for some $C^\infty$ function $f$. We proved in class that $i \int_X \alpha \wedge \tilde{\alpha} \geq 0$, and if this integral vanishes, then necessarily $\alpha = 0$ (this also follows from the Hodge index theorem but we did it using more elementary methods). But

$$\int_X \alpha \wedge \tilde{\alpha} = \int_X \alpha \wedge \bar{\partial} f = - \int_X d(f \alpha) = 0.$$

Thus the map is injective.
(e) Deduce from Serre duality that it is also surjective and that we have the decomposition

\[ H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)} \]

with \( \overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O}_X) \).

By Serre Duality, \( H^1(X, \mathcal{O}_X) \cong H^0(X, \Omega_X)^* \) since now \( K_X \cong \Omega_X \). It follows that \( \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X)^* = \dim H^0(X, \Omega_X) \), and so by the previous problem, we have an injective map between vector spaces with the same dimension. This map should also be surjective by linear algebra. As a consequence, \( \overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O}_X) \).

Now the complex conjugate of the short exact sequence is identified with

\[ 0 \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathbb{C}) \to H^0(X, \Omega_X) \to 0. \]

This sequence gives a splitting of the original short exact sequence. It follows that

\[ H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}. \]