Questions

1. Using Theorem 1.28, and the argument given in the proof of proposition 2.31, show that any form of type $(0, i)$ which is $\bar{\partial}$-closed in $\mathbb{C}^n$ is $\bar{\partial}$-exact for $i > 0$. Deduce from this that

$$H^i(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0, \quad \forall i > 0$$

First, we prove the Poincaré lemma for the polydisk $B_l = \{|z| < l, 1 \leq i \leq n\}$. Let $\alpha = \sum_{|J|=\gamma} \alpha_J d\bar{z}_J$ be the $\bar{\partial}$-closed $(0, i)$-form and we show that there’s a $\beta \in \mathcal{O}(B_l)$ such that $\bar{\partial} \beta = \alpha$. In fact, we can prove this for $l = 1$ and the other disks comes from the scaling $z \mapsto \frac{z}{l}$.

We do the induction on the highest index involved in $\alpha$, call it $k$. Then we first write the $(0, i)$-form $\alpha$ in the form $\alpha = \beta \wedge d\bar{z}_k + \alpha'$, where $\alpha'$ only involves $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$. If $k = i$, then $\alpha = f d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_k$ and so $f$ is holomorphic in $z_l$ for $l > k$ since $\bar{\partial} \alpha = 0$. So that locally, we set

$$g = \frac{1}{2\pi i} \int \frac{f(z_1, \ldots, w_k, z_{k+1}, \ldots, z_n)}{w_k - z_k} dw_k \wedge d\bar{w}_k$$

This integral is over the whole $\mathbb{C}$ and we extend $f$ by a bump function on the disk. By Cauchy’s integral formula, for all the points inside the disk $|z| < l$, we have $\frac{\partial g}{\partial \bar{w}_k}(\bar{z}_k, w_k) = f$, moreover $\frac{\partial g}{\partial \bar{w}_k}(z_k, w_k) = 0$ for $l > k$ since $f$ is holomorphic on $z_l$ for $l > k$ and so we have

$$(-1)^{-i-1} \bar{\partial}(gd\bar{z}_1 \wedge \ldots \wedge d\bar{z}_{k-1}) = f d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_k$$

Now we do the case $k > i$. Since we can write $\beta = \sum f_J d\bar{z}_J$ and $J \subset \{1, 2, \ldots, k-1\}$, by a similar argument $f_J$ are holomorphic in $z_l$ for $l > k$. So if we set

$$g_J(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|w_k| < 1} \frac{g_J(z_1, \ldots, w_k, \ldots, z_{n-1}, z_n)}{w_k - z_k} dw_k \wedge d\bar{w}_k$$

By the Cauchy integral formula, $\frac{\partial g_J}{\partial \bar{w}_k} = f_J$. We set

$$\gamma = (-1)^{-i-1} \sum g_J d\bar{z}_J.$$

We get

$$\bar{\partial} \gamma = \sum f_J d\bar{z}_J \wedge d\bar{z}_k + (-1)^{-i-1} \sum_{l<k} \frac{\partial f_J}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_J.$$

So $\alpha = \bar{\partial} \gamma + \alpha''$, where $\alpha''$ is $\bar{\partial}$-closed and only involves $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$, so by induction $\alpha''$ is $\bar{\partial}$-exact. It follows immediately that $\alpha$ is $\bar{\partial}$-exact.

We let $\mathbb{C}^n = \bigcup_{m \in \mathbb{N}} B_m = B_1 \cup B_2 \cup \ldots$. We first prove that for any $m$ there’s a $(0, i-1)$-form $\beta_m$ on $\mathbb{C}^n$ such that $\bar{\partial} \beta_m = \alpha$ on $B_m$. By the above discussion we can find a form on $\beta'_m$ on $B_{m+1}$ with $\bar{\partial} \beta'_m = \alpha$ on $B_m$. We choose a support function $\psi_m$ such that $\psi_m$ is differentiable and $\psi_m = 1$ on $B_m$, $\psi_m = 0$ on $\mathbb{C}^n \backslash B_{m+1}$. Now $\psi_m \beta'_m$ gives us the desired form. Thus $\bar{\partial} \beta_m = \bar{\partial} \beta'_m = \alpha$ on $B_m$.

Now we claim that we can find a sequence $\{\beta_m\}$ such that $\beta_m = \beta_{m+1}$ on $\overline{B_{m+1}}$.

If $i > 1$ we do this by induction. Suppose we already choose $\beta_1, \ldots, \beta_m$, and we know that there’s a $\bar{\partial}(\beta_{m+1}) = \alpha$ on $B_m$ and so $\bar{\partial}(\beta_m - \beta_{m+1}) = 0$ on $B_m$. This will yield $\beta_m - \beta_{m+1} = \bar{\partial} \gamma$ on $B_m$ where $\gamma$ is a $(0, i-2)$-form. Now set $\psi_{m-1}$ to be the support function defined before, and we let $\beta_{m+1} = \beta_{m+1} + \bar{\partial}(\psi_{m-1} \gamma)$. We check that $\beta_{m+1} = \bar{\partial}(\gamma) = \beta_m$ on $\overline{B_{m+1}}$ and
\( \partial \beta_{m+1} = \bar{\partial} \beta_{m+1} = \alpha \) on \( B_{m+1} \), as desired. Thus the sequence \( \{ \beta_m \} \) converges to a form \( \beta \) which gives \( \bar{\partial} \alpha \) globally on \( \mathbb{C}^n \).

If \( i = 1 \), we'll need to construct a sequence of functions \( \{ \beta_m \} \) such that \( \bar{\partial} \beta_m = \alpha \) on \( B_m \) and \( |\beta_{m+1} - \beta_m| < 2^{-m} \) on \( \overline{B_{m-1}} \) and so this will gives a uniform convergent sequence of function. If we have constructed already \( \beta_1, \ldots, \beta_m \), and by previous argument we can construct \( \bar{\partial} \beta = \alpha \) on \( B_{m+1} \). We see that \( \bar{\partial}(\beta_m - \beta_{m+1}) = 0 \) on \( \overline{B_{m-1}} \), and so \( f_m = \beta_m - \beta_{m+1} \) is a holomorphic function on \( \overline{B_{m-1}} \). We look at the Taylor series of \( f_m \) and we can find a polynomial \( P_m \) such that \( |f_m - P_m| < 2^{-m} \) on \( \overline{B_{m-1}} \) by the truncation of the series.

The Dolbeault resolution gives the cohomology of \( \mathcal{O}_{\mathbb{C}^n} \) as

\[
H^i(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = \frac{\ker(\bar{\partial} : A^{0,i} \to A^{0,i+1})}{\text{im}(\partial : A^{0,i-1} \to A^{0,i})}
\]

In particular, if \( i \geq 1 \), \( H^i(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \) is the \( \bar{\partial} \)-closed \((0,i)\)-forms modulo the \( \bar{\partial} \)-exact \((0,i)\)-forms. Since we’ve proven that every \( \bar{\partial} \)-closed \((0,i)\)-form on \( \mathbb{C}^n \) is exact for \( i \geq 1 \), we have \( H^i(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0 \).

2. Let \( X \) be a compact complex curve. Let \( \mu \) be a volume form on \( X \) and we can consider \( \mu \) as a closed form of type \((1,1)\) on \( X \).

(a) By considering the integral \( \int_X \mu \) show that \( \mu \) is not \( \bar{\partial} \)-exact. Deduce from this that \( H^1(X, K_X) \) is different from \( \{0\} \) and admits a surjective map

\[
\text{Tr} : H^1(X, K_X) \to \mathbb{C}, \quad \omega \mapsto \int_X \omega.
\]

Since \( \mu \) is a volume form, \( \int_X \mu \neq 0 \). Also \( \bar{\partial} \mu = 0 \) since all \((1,2)\)-forms on a complex curve vanish. And \( \mu \) is not \( \bar{\partial} \)-exact since on the contrary, that would imply \( \int_X \mu = \int_X \bar{\partial} \alpha = \int_X d\alpha = \int_{\partial X} \alpha = 0 \) since \( X \) is compact and has no boundary.

We see that this \( \mu \) determines a class \([\mu]\) in

\[
H^1(X, \Omega^1_X) = \frac{\ker(\bar{\partial} : A^{1,1} \to A^{1,2})}{\text{im}(\partial : A^{1,0} \to A^{1,1})}
\]

where \( \Omega^1_X \) is the line bundle of holomorphic 1-forms. Since \( K_X = \Omega^1_X \), we see that \( H^1(X, K_X) \neq \{0\} \). We have also computed above that \( \text{Tr}([\mu]) = \int_X \mu \neq 0 \). Then given any \( z \in \mathbb{C} \) we have \( \text{Tr}([z/(\text{Tr}([\mu]) \mu)]) = z \), and \( \text{Tr} \) is surjective.

(b) Show that we have an exact sequence

\[
0 \to K_X \to K_X(D) \xrightarrow{\text{Res}_i} \sum_i \mathbb{C}_{x_i} \to 0
\]

where each of the \( \mathbb{C}_{x_i} \) is a “skyscraper” sheaf supported at \( x_i \). Here the map \( \text{Res}_i : K_X(D) \to \mathbb{C} \) maps a meromorphic form \( \omega \) to its residue at \( x_i \), defined as

\[
\text{Res}_i(\omega) = \int_{\partial D_i} \frac{1}{2i\pi} \omega
\]

To show the exactness, we just need to show it on stalks. This means that for every \( p \), \( 0 \to (K_X)_p \to (K_X(D))_p \xrightarrow{\text{Res}_i} (\sum_i \mathbb{C}_{x_i})_p \to 0 \) is an exact sequence.

If \( p \neq x_i \), then \( (K_X(D))_p = (K_X)_p \) and \( (\sum_i \mathbb{C}_{x_i})_p = 0 \), and exactness follows immediately.

If \( p = x_i \) for some \( x_i \), we have \( (\sum_i \mathbb{C}_{x_i})_p = \mathbb{C}_{x_i} \), and we need to prove \( 0 \to (K_X)_x \to (K_X(D))_{x_i} \xrightarrow{\text{Res}_x} \mathbb{C} \to 0 \) is an exact sequence.
The injectivity of $(K_X)_{x_i} \rightarrow (K_X(D))_{x_i}$ is clear, since every germ of a nonzero holomorphic 1-form will go to the same germ. The surjectivity of $(K_X(D))_{x_i} \rightarrow \mathbb{C}$ is done by looking at $\text{Res}(\phi(z_i)\frac{dz_i}{z_i})$ and we know by standard complex analysis that the residue will be $2i\pi \phi(x_i)$. So we can choose $\phi(x_i)$ to be any complex number to get a surjection onto $\mathbb{C}$.

To prove that $\ker \text{Res}_{x_i} = (K_X)_{x_i}$, we first notice that $\text{Res}_{x_i}(\omega) = 0$ if $\omega \in (K_X)_{x_i}$.

Now if we have a $\omega \in (K_X(D))_{x_i}$, it can locally be written as $\phi(z_i)\frac{dz_i}{z_i}$ in a sufficiently small neighborhood of $x_i$. If we suppose $\text{Res}_{x_i}\omega = 0$, it means $\phi_i(x_i) = 0$. Since $\phi_i(z_i)$ is holomorphic, $\phi_i(z_i) = z_i \psi_i(z_i)$ for a holomorphic $\psi_i(z_i)$, which means $\omega = \psi_i(z_i)dz_i$ locally and is naturally inside $(K_X)_{x_i}$.

(c) Let $\delta : \oplus_i \mathbb{C} = H^0(X, \oplus_i \mathbb{C}) \rightarrow H^1(X, K_X)$ be the arrow appearing in the long exact sequence associated to the short exact sequence. Show that $\delta(1_{x_i})$ is the class in $H^1(X, K_X)$ of the form $\partial \mu_i$, where $\mu_i$ is a differential form of type $(1, 0)$, which is $C^\infty$ away from $x_i$ and equal to $dz_i/z_i$ in a neighborhood of $x_i$.

We look at the commutative diagram associated to the exact sequence

$$
0 \rightarrow A^{1,0}_X \rightarrow A^{1,0}_X(D) \xrightarrow{\text{Res}} \sum_i \mathbb{C}_{x_i} \rightarrow 0
$$

which is a resolution (in the downward direction) of the short exact sequence in (b) above. Let $\mu_i$ be such a $(1, 0)$ form, which is a global section of $A^{1,0}_X(D)$. Then $\text{Res}(\mu_i)$ is the element of $\sum_i \mathbb{C}_{x_i}$ with component 1 in $\mathbb{C}_{x_i}$ and 0 in $\mathbb{C}_{x_j}$ for $j \neq 0$. By the standard definition of $\delta$ by a diagram chase, we know that $\partial \mu_i$, a priori a global section of $A^{1,1}_X(D)$, is actually a global section of $A^{1,1}_X$, is $\partial$-closed, and represents $\delta(1_{x_i})$ in $H^1(X, K_X)$ identified with $\partial$-cohomology.

(d) Show that $\int_X \partial \mu_i = -2i\pi$. Deduce from the long exact sequence associated to the short exact sequence the following result:

If $\omega$ is a meromorphic 1-form on $X$ having poles of order at most 1 at each $x_i$, and holomorphic otherwise, then

$$
\sum_i \text{Res}_i \omega = 0.
$$

We let $U_c = X \setminus B_c(x_i)$, then $\int_{U_c} \partial \mu_i = \int_{\partial U_c} \mu_i = -\int_{B_c(x_i)} \mu_i = -2i\pi$. We then evaluate the improper integral by $\int_X \partial \mu_i = \lim_{r \to 0} \int_{U_r} \partial \mu_i = -2i\pi$.

By the long exact sequence we have $\delta \circ \text{Res}(\omega) = 0$. On the other hand, since $\text{Res}(\omega) = \text{Res}_{x_1}(\omega)1_{x_1} + \ldots + \text{Res}_{x_n}(\omega))1_{x_n}$, we know that $\delta \circ \text{Res}(\omega) = \sum_i \text{Res}_{x_i}(\omega)\partial \mu_i$. By integrating over $X$, we see

$$
0 = \int_X \delta \circ \text{Res}(\omega) = \int_X \sum_i (\text{Res}_i(\omega))\partial \mu_i = -2i\pi \sum_i (\text{Res}_i(\omega))
$$

3. Let $\tau \in \mathbb{C}$ and let $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, prove that $\dim H^1(E_\tau, \mathcal{O}_{E_\tau}) = H^0(E_\tau, \Omega^1_{E_\tau}) = 1$.

For any holomorphic $(1,0)$-form $\alpha$, we claim that $\alpha = c \cdot dz$ and $c$ is a constant function. This is because locally we can write $\alpha = f(z)dz$ and the transition function will be $f(z)dz \rightarrow g(w)dw$, where $w = z + n \cdot g(w) = f(z - n)$. Thus $f$ defines a holomorphic function on $E_\tau$. Since every holomorphic function on a compact complex manifold is constant, we know that $f$ is a constant function. As a result, $\alpha = c \cdot dz$ and $\partial \alpha = 0$ since $c$ is a constant.
This shows that $H^0(E^r, \Omega^1_{E^r}) = \mathbb{C} dz$, hence is 1-dimensional.

To show that $\dim H^1(E^r, \mathcal{O}_{E^r}) = 1$, we use Fourier analysis. First of all, every $C^\infty$ function $f$ on $E^r$ is a doubly periodic function on $\mathbb{C}$ with periods 1 and $\tau$, which by standard Fourier analysis, can be written in the form

$$f(z) = \sum_{(n,m) \in \mathbb{Z}^2} Q_{n,m}(f) e^{2\pi i (a_n z + b_m \bar{z})}.$$  

where the constants $a_{n,m}$ and $b_{n,m}$ need to be determined. We define the $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$ by $T(z) = a_{n,m} z + b_m \bar{z}$ and require $T(1) = n$ and $T(\tau) = m$, so that $e^{2\pi i T(z)}$ will be the Fourier basis since $e^{2\pi i T(z+1)} = e^{2\pi i T(z)}$. The condition implies $a_{n,m} \cdot 1 + b_m \cdot 1 = n$, $a_{n,m} \cdot \tau + b_m \cdot \bar{\tau} = m$. As a result $a_{n,m} = \frac{n \tau - m}{\bar{\tau} - \tau}$ and $b_m = \frac{n \tau - m}{\bar{\tau} - \tau}$. We set $f_{n,m} = e^{2\pi i (a_{n,m} z + b_m \bar{z})}$ as the basis for doubly periodic function.

We just need to see how $\bar{\partial}$ acts on $f_{n,m}$ in $\mathcal{A}^{0,0}$. We compute that $\bar{\partial} f_{n,m} = 2\pi i b_m f_{n,m} d\bar{z}$. Since 1 and $\tau$ are linearly independent over $\mathbb{R}$, we see that $b_{n,m} = 0$ if and only if for $(n,m) = (0,0)$. It follows that

$$f(z) d\bar{z} = Q_{0,0}(f) d\bar{z} + \bar{\partial} \left( \sum_{(n,m) \neq (0,0)} \frac{Q_{n,m}}{2\pi i b_{n,m}} f_{n,m} \right)$$

and this series converges by standard Fourier analysis.

Now we’ve proven that every $f d\bar{z} = cd\bar{z} + \bar{\partial}(g)$, so that in the Dolbeault resolution $0 \to \mathcal{A}^{0,0} \to \mathcal{A}^{0,1} \to 0$, the first cohomology $H^1(X, \mathcal{O}_X) = \mathcal{A}^{1,0}/\bar{\partial}\mathcal{A}^{0,0}$ has only the class generated by $d\bar{z}$. The above calculation also shows that $d\bar{z}$ is not $\bar{\partial}$-exact, hence represents a nonzero element of $H^1(E^r, \mathcal{O}_{E^r})$.

4. Let $\mathcal{S}$ be the locally free sheaf of sections of the universal line bundle $\mathcal{S}$ over $\mathbb{P}^n$. Let $\mathcal{O}_{\mathbb{P}^n}(-1)$ be the sheaf of homogeneous expressions of degree $-1$. More precisely, for any open set $U \subset \mathbb{P}^n$ we define

$$\mathcal{O}_{\mathbb{P}^n}(-1)(U) = \{ f(x_0, \ldots, x_n) \in \mathcal{O}(\pi^{-1}(U)) | f(\lambda x_0, \ldots, \lambda x_n) = \lambda^{-1} f(x_1, \ldots, x_n) \},$$

where $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ is the natural projection map.

Show that $\mathcal{S}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

We first find a map between $\mathcal{S}$ and $\mathcal{O}_{\mathbb{P}^n}(-1)$, then show that it’s an isomorphism.

A section of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $U$ is a holomorphic function on $\pi^{-1}(U)$ such that $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^{-1} f(x_1, \ldots, x_n)$. The space of sections of $\mathcal{S}$ on an open subset $U$ of $\mathbb{P}^n$ is defined to be

$$\mathcal{S}(U) = \left\{ (g(x_1, \ldots, x_n)) : U \to \mathbb{C}^{n+1} \ | \ g(x_0, \ldots, x_n) = k \cdot (x_1, \ldots, x_n) \text{ for some } k \in \mathbb{C}, \right. \left. g(\lambda x_0, \ldots, \lambda x_n) = g(x_1, \ldots, x_n) \right\}$$

Note that the argument $(x_0, \ldots, x_n)$ of $g \in \mathcal{S}(U)$ represents homogeneous coordinates in $\mathbb{P}^n$ while the argument $(x_0, \ldots, x_n)$ of $f \in \mathcal{O}_{\mathbb{P}^n}(-1)(U)$ represents coordinates in $\pi^{-1}(U)$.

Define $\phi : \mathcal{O}_{\mathbb{P}^n}(-1)(U) \to \mathcal{S}$, $\phi(f) = (f(x_0, \ldots, x_n))(x_0, \ldots, x_n)$. This mapping is well-defined: Since $f(\lambda x_0, \ldots, \lambda x_n) \cdot (\lambda x_0, \ldots, \lambda x_n) = f(x_0, \ldots, x_n) \cdot (x_0, \ldots, x_n)$, we see that $\phi(f)$ sits inside $\mathcal{S}(U)$. The mapping $\phi$ is compatible with restriction, so defines a mapping of sheaves.

We now show that $\phi$ is an isomorphism.

For every $g \in \mathcal{S}(U)$, we define $\psi(g) \in \mathcal{O}_{\mathbb{C}^{n+1}}(\pi^{-1}(U))$ by $g(x_0, \ldots, x_n) = ((\psi(g))(x_0, \ldots, x_n))(x_0, \ldots, x_n)$. $\psi(g)$ is a holomorphic function since $\mathcal{S}$ is a holomorphic bundle. Moreover, $\psi(g)(\lambda x_0, \ldots, \lambda x_n) = \lambda^{-1} \psi(g)(x_0, \ldots, x_n)$ since $g(x_0, \ldots, x_n) = g(\lambda x_0, \ldots, \lambda x_n)$. So $\psi(g) \in \mathcal{O}_{\mathbb{P}^n}(-1)(U)$ and satisfies $\psi(f) = f$, $\phi(\psi(g)) = g$.

This tells us that $\phi$ and $\psi$ defines an isomorphism between $\mathcal{S}(U)$ and $\mathcal{O}_{\mathbb{P}^n}(-1)(U)$. Since this works for all open subset $U \subset \mathbb{P}^n$, it’s an isomorphism between $\mathcal{S}$ and $\mathcal{O}_{\mathbb{P}^n}(-1)$. 
