Questions

1. Let $X$ be a connected complex manifold, and $h$ be a Kähler metric. Let $\omega$ be the associated Kähler form. Show that if $\dim X \geq 2$, the wedge product with $\omega$ is injective on 1-forms.

Show that if $\dim X \geq 2$ and $\phi$ is a differentiable function with values in $\mathbb{R}^+$ such that $\phi h$ is also a Kähler metric, then is constant.

Consider a 1-form $\alpha$ such that $\omega \wedge \alpha = 0$. We’ll show that $\alpha = 0$. Actually, it suffices to prove it for a $(1,0)$ form $\alpha$, since we consider locally that $\alpha = \sum a_j dz_j + \sum b_k d\bar{z}_k$ and $\omega = \sum \omega_{jk} dz_j \wedge d\bar{z}_k$, now we have

$$\omega \wedge \alpha = \sum_{j,k,l} a_{jk} \omega_{kl} dz_j \wedge dz_k \wedge d\bar{z}_l + \sum_{j,k,l} b_{j,k} \omega_{kl} dz_k \wedge d\bar{z}_l \wedge d\bar{z}_j.$$ 

But since $dz_j \wedge dz_k \wedge d\bar{z}_l$ is a $(2,1)$-form and $dz_k \wedge d\bar{z}_l \wedge d\bar{z}_j$ is a $(1,2)$-form, they won’t cancel with each other, thus we just need to prove it for $\alpha = \sum a_j dz_j$ and the proof for $\beta = \sum b_j d\bar{z}_j$ is similar.

Now $\omega \wedge \alpha = \sum_{j,k,l} a_{jk} \omega_{kl} dz_j \wedge dz_k \wedge d\bar{z}_l$, and we suppose on the contrary that $a_m \neq 0$ for some $m$ in a neighborhood of some point. Now we look at the coefficient of $dz_m \wedge dz_k \wedge d\bar{z}_l$. If $m = k$, this is automatically 0 since $dz_m \wedge dz_m = 0$. So we only need to consider $m \neq k$, and now the coefficient will be $a_m \omega_{kl} - a_k \omega_{ml}$. This implies $\omega_{kl} = \frac{a_k}{a_m} \omega_{ml} \forall l$. It follows that

$$\omega = \frac{1}{a_m} \left( \left( \sum_k a_k dz_k \right) \wedge \left( \sum_{m,l} \omega_{ml} d\bar{z}_l \right) \right)$$

Thus $\omega \wedge \omega = 0$, which implies that $\omega^{\dim X} = 0$, contradicting the nondegeneracy of $\omega$.

If $\phi \omega$ is also a Kahler form, then $d(\phi \omega) = 0$ and this gives $d\phi \wedge \omega = 0$ since $d\omega = 0$. By the first part we conclude that $d\phi = 0$. Hence $\phi$ is a constant function.

2. Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. Show $L$ is a holomorphic line bundle on $X$, $\mathbb{P}(E^* \otimes L^*) \cong \mathbb{P}(E^*)$, but the line bundle $\mathcal{O}_{\mathbb{P}(E^* \otimes L^*)}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L$, where $\pi : \mathbb{P}(E^*) \to X$ is the structural projection.

We consider local trivializations of $E^*$ and $L$. We can choose a covering $\{U_j\}$ of $X$ with gluing given by

$$(z,v) \in (U_j \cap U_k) \times \mathbb{C}^r \subset U_j \times \mathbb{C}^r \mapsto (z, \psi_{jk}(z) \cdot v) \in U_k \times \mathbb{C}^r.$$ 

In the above, $\psi_{jk}(z) \in GL_r(\mathcal{O}(U_j \cap U_k))$, that is, $\psi_{jk}$ are holomorphic functions with values in general linear group $GL_r(\mathbb{C})$. Analogously, we can define the transition functions of $L$ from $U_j$ to $U_k$ to be $\rho_{jk}$, where $\rho_{jk}$ takes value in $\mathbb{C}^*$. Now by definition $\mathbb{P}(E^*)$ has a local trivialization $U_j \times \mathbb{P}^{r-1}$. Letting $[v] = [v_0, \ldots, v_{r-1}]$ denote homogeneous coordinates in $\mathbb{P}^{r-1}$, the transition map is

$$(z,[v]) \mapsto (z,[\psi_{jk}(z)^t v]).$$

On the other hand, the transition map for $\mathbb{P}(E^* \otimes L^*)$ is

$$(z,[v_j]) \mapsto (z,[\rho_{jk}(z)^{-1} \psi_{jk}(z)^t v]).$$

But in $\mathbb{P}^{r-1}$ we have $[\psi_{jk}(z)v_j] = [\rho_{jk}(z)^{-1} \psi_{jk}(z)^t v]$ for all $z \in U_j \cap U_k$ as $\rho_{jk}(z)^{-1} \psi_{jk}(z)^t v$ is a scalar multiple of $\psi_{jk}(z)^t v$ by the nonzero complex number $\rho_{jk}(z)^{-1} \in \mathbb{C}^*$. Thus, the identification
of each $U_j \times \mathbb{P}^{r-1} \subset \mathbb{P}(E^*)$ with $U_j \times \mathbb{P}^{r-1} \subset \mathbb{P}(E^* \otimes L^*)$ is compatible with the transition matrices. Hence these identifications patch to give a canonical bijection $\phi : \mathbb{P}(E^*) \rightarrow \mathbb{P}(E^* \otimes L)$.

To see that $\phi$ is holomorphic, we cover each $\mathbb{P}(E^*)$ and $\mathbb{P}(E^* \otimes L^*)$ by the inverse images of $U_j$ under the respective projections, each isomorphic to $U_j \times \mathbb{P}^{r-1}$. Then $\phi$ maps $U_j \times \mathbb{P}^{r-1}$ to $U_j \times \mathbb{P}^{r-1}$, and the above explicit description of $\phi$ show that with these local identification, $\phi$ is identified with the identity map on $U_j \times \mathbb{P}^{r-1}$, which is clearly holomorphic. Hence $\phi$ is holomorphic.

To show that $\phi$ identifies $\mathcal{O}_{\mathbb{P}(E^* \otimes L^*)}(1)$ with $\mathcal{O}_{\mathbb{P}(E^*)}(1) \otimes \pi^* L$, it is equivalent to compare their duals, which amounts to comparing $S_{E^* \otimes L^*}$ and $S_{E^*} \otimes \pi^* L^*$, where $S_{E^* \otimes L^*}$ and $S_{E^*}$ are respectively the tautological subbundles on $\mathbb{P}(E^* \otimes L^*)$ and $\mathbb{P}(E^*)$.

Over $U_j$, $S_{E^* \otimes L^*}$ is locally described as the set of tuples $(z, [v], lv) \in U_j \times \mathbb{P}^{r-1} \times \mathbb{C}^*$, and the transition map is $(z, [v], lv) \rightarrow (z, [\rho_{jk}(z)^{-1}\psi_{jk}(z)v], l\rho_{jk}(z)^{-1}\psi_{jk}(z)v)$.

Similarly, the transition map for $S_{E^*}$ is given by $(z, [v], lv) \rightarrow (z, [\psi_{jk}(z)v], l\psi_{jk}(z)v)$. Since the transition map for $\pi^* L^*$ is $(z, [v], l) \rightarrow (z, [\psi_{jk}(z)v], l\rho_{jk}(z)^{-1})$, then after we tensor $S_{E^*}$ with $\pi^* L^*$, the transition map will be $(z, [v], lv) \rightarrow (z, [\psi_{jk}(z)v], l\rho_{jk}(z)^{-1}\psi_{jk}(z)v)$, which is exactly the same as the transition function of $S_{E^* \otimes L^*}$ since $[\psi_{jk}(z)v] = [\rho_{jk}(z)^{-1}\psi_{jk}(z)v]$.

3. Prove that any compact Riemann Surface $X$ is Kähler.

Let $h$ be any Hermitian metric on $X$, and $\omega$ the associated $(1,1)$ form. Since there are no nonzero 3-forms on $X$, it follows that $d\omega = 0$ and $X$ is Kähler.

We have shown more: that any Hermitian metric on $X$ is a Kähler metric.

4. Prove the formula

$$d\omega(\phi, \chi, \psi) = \phi(\omega(\chi, \psi)) - \chi(\omega(\phi, \psi)) + \psi(\omega(\phi, \chi)) - \omega(\phi, \chi)) \omega + \omega([\phi, \chi], \psi) + \omega([\phi, \psi], \chi)$$

First, we verify that the right hand side is multilinear in $\phi, \chi$ and $\psi$ over the ring of $C^\infty$ functions. We let $I = \phi(\omega(\chi, \psi)) - \chi(\omega(\phi, \psi)) + \psi(\omega(\phi, \chi))$ and $II = -\omega([\phi, \chi], \psi) + \omega([\phi, \psi], \chi)$.

Since $I$ and $II$ are each alternating in $\phi, \chi, \psi$, it suffices to verify linearity in $\phi$ over $C^\infty$ functions.

For $I$, we have

$$f\phi(\omega(\chi, \psi)) - \chi(\omega(f\phi, \psi)) + \psi(\omega(f\phi, \chi)) = f(\phi(\omega(\chi, \psi)) - \chi(\omega(\phi, \psi)) + \psi(\omega(\phi, \chi))) - \chi(f)(\omega(\phi, \psi)) + \psi(f\omega(\phi, \chi))$$

For $II$, we get

$$-\omega([f\phi, \chi], \psi) = -\omega(f\phi\chi - \chi(f\phi), \psi) = -f\omega(\phi\chi - \chi(f\phi), \psi) = -f\omega([\phi, \chi], \psi) + \chi(f)(\omega(\phi, \psi));$$

$$\omega(f\phi, [\chi, \psi]) = f\omega(\phi, [\chi, \psi]);$$

$$\omega([f\phi, \psi], \chi) = f\omega([\phi, \psi], \chi) - \psi(f)(\omega(\phi, \chi))$$

This cancels exactly the last two terms in $I$ and leaves $f$ times the original expression. Thus $I + II$ is linear with respect to any differentiable function.

Since the $\partial / \partial x_i$ generate all vector fields over the ring of $C^\infty$ functions, it suffices to show the identity for all choices of triples of the $\partial / \partial x_i$. We consider $\phi = \frac{\partial}{\partial x_i}, \chi = \frac{\partial}{\partial x_j}, \psi = \frac{\partial}{\partial x_k}$ and let $\omega = \sum h_{ijk} dx_j \wedge dx_k$. We may assume $j, k, l$ are different since otherwise we get $0$. Now $d\omega = \sum_{j,k,l} \frac{\partial h_{ijk}}{\partial x_l} dx_j \wedge dx_k \wedge dx_l$ and so the left hand side will be

$$\frac{\partial h_{ijk}}{\partial x_l} - \frac{\partial h_{ijl}}{\partial x_k} + \frac{\partial h_{klj}}{\partial x_j}.$$
We must see what’s on the right hand side. All the terms in $II$ give us 0 since $\phi, \chi$ and $\psi$ commute with each other. For $I$, we get exactly

$$\frac{\partial}{\partial x_j} h_{jk} - \frac{\partial}{\partial x_k} h_{jl} + \frac{\partial}{\partial x_l} h_{jk}$$

which is exactly the same as what we get on the left hand side.