HOMEWORK 2: Complex Manifolds

Questions

1. Let $X$ be a complex manifold and $L$ be a holomorphic line bundle on $X$. We assume that there exist a holomorphic line bundle $K$ on $X$ and an isomorphism $K^\otimes 2 \cong L$. Assume we are given a non-zero holomorphic section $\sigma$ of $L$. We denote by $\Sigma \subset L$ the image of $\sigma$.

(a) Show that the map

$$\phi : K \to L, \quad (x, \tau) \to (x, \tau^2)$$

is a proper holomorphic map.

If this map is well defined, then it is holomorphic since locally it’s the map $(x, \tau) \to (x, \tau^2)$, and we know that the identity map as well as $f(\tau) = \tau^2$ are holomorphic. Thus it suffices to prove that this map is well-defined.

Let $\kappa_{jk}$ be the transition functions for $K$ relative to an open cover \{U_j\}, so that the $\kappa_{jk}$ are holomorphic functions on $U_j \cap U_k$. Since $K^\otimes 2 \cong L$, we can take $\lambda_{jk} = \kappa_{jk}^2$ as the transition functions for $L$. Locally over $U_j$, we can identify $K$ and $L$ with $U_j \times \mathbb{C}$. Using these local trivializations, we write $\phi(x, \tau) = (x, \tau^2)$ and check that these local descriptions glue together properly. We have to show that the following diagram commutes

$$\begin{array}{ccc}
U_j \cap U_k \times \mathbb{C} & \xrightarrow{(\text{id}, \kappa_{jk})} & U_j \cap U_k \\
\phi \downarrow & & \downarrow \phi \\
U_j \cap U_k \times \mathbb{C} & \xrightarrow{(\text{id}, \lambda_{jk})} & U_j \cap U_k \times \mathbb{C}
\end{array}$$

which is easily checked.

To prove that $\phi$ is proper, it suffices to prove that the preimage of a compact set $C \subset L$ is compact. We choose a sequence $\{y_n\}$ in $\phi^{-1}(C)$ and show that it has a limit point in $\phi^{-1}(C)$.

We have the sequence $\{\phi(y_n)\}$ in $C$. By compactness of $C$, we can replace the $\{y_n\}$ by an infinite subsequence such that $\{\phi(y_n)\}$ converges, and each of the $\phi(y_n)$ as well as its limit lie in one of the open sets in our local trivialization of $L$, which we identify with $U_i \times \mathbb{C}$. Writing $\phi(y_n) = (x_n, \rho_n) \in U_i \times \mathbb{C}$, we have a limit $\{(x_n, \rho_n)\} \to (x, \rho) \in C \cap (U_i \times \mathbb{C})$. Then the sequence $\{\rho_n\}$ is bounded. Since $y_n = (x_n, \pm \sqrt{\rho_n})$ for some choice of square root, the sequence $\pm \sqrt{\rho_n}$ is also bounded, hence has a convergent subsequence. This gives a corresponding convergent subsequence of $\{y_n\}$, which must converge to $(x, \pm \sqrt{\rho})$ for one of the two square roots of $\rho$.

Since $\phi(x, \pm \sqrt{\rho}) = (x, \rho) \in C$, we have that $(x, \pm \sqrt{\rho}) \in \phi^{-1}(C)$, as required.

(b) Show that $Y = \phi^{-1}(\Sigma)$ is smooth if and only if $R$ is smooth.

We begin by looking at the definition of $Y$. Let the dimension of $X$ be $n$, and on patch $U_j$ we have local coordinates $(x_1, \ldots, x_n)$. So on this patch, the section $\sigma$ is defined to be a function of $(x_1, \ldots, x_n)$. As a result, $Y$ is defined to be the zero locus of $f : U_j \times \mathbb{C} \to \mathbb{C}$ with

$$f(x_1, \ldots, x_n, \tau) = \tau^2 - \sigma(x_1, \ldots, x_n).$$

$Y$ is well defined inside $K$ with the same argument in previous part, thus we need to see when $f^{-1}(0)$ smooth.

We need to find smoothness on the given point $p_0 = (z_1, \ldots, z_n, \tau_0) \in f^{-1}(0)$. This is divided into 2 cases:
(1) If $\sigma(z_1, \ldots, z_n) \neq 0$, we get $\tau_0 \neq 0$, thus $\frac{\partial f}{\partial \tau} = 2\tau$ is nonzero on a small neighborhood of $p_0$. By the implicit function theorem, $\tau$ is locally a holomorphic function of $(x_1, \ldots, x_n)$ and thus $f^{-1}(0)$ is smooth near $p_0$, since it’s just the graph of a holomorphic function.

(2) If $\sigma(z_1, \ldots, z_n) = 0$ we know that $\tau_0 = 0$ and we can’t use the implicit function theorem for $\tau$ as the partial derivative $2\tau$ is not invertible near $\tau_0$.

We should now look at all $\frac{\partial f}{\partial x_l}$, for $1 \leq l \leq n$. If $\frac{\partial f}{\partial x_l}(z_1, \ldots, z_n) = 0$ for all $l$, then $p_0 = (z_1, \ldots, z_n)$ is the singularity for $R$ and $f^{-1}(0)$ is singular at $p_0$ as well.

On the other hand, if $p_0$ is not a singularity, there exist one $m$ such that $\frac{\partial \sigma}{\partial x_m}(z_1, \ldots, z_n) \neq 0$. Now the Jacobian $\frac{\partial \sigma}{\partial x_m}$ is nonzero near a neighborhood of $p_0$. By using implicit function theorem on $x_m$ we know that $x_m$ is locally a holomorphic function of $(x_l)_{l \neq m}$ and $\tau$, and thus $f^{-1}(0)$ is smooth at $p_0$.

(c) Show that when $R$ is smooth, $\phi : Y \to \Sigma \cong X$ is ramified exactly along $\phi^{-1}(R) \cong R$. Show that the fibers $\phi^{-1}(x)$ consists of 2 distinct points when $x \notin R$.

The image $\Sigma$ is locally equal to $g^{-1}(0)$, where $g : U_j \times \mathbb{C} \to \mathbb{C}$ by $g(x_1, \ldots, x_n, \rho) = \rho - \sigma(x_1, \ldots, x_n)$.

$\phi^{-1}(R) \cong R$ since $\phi^{-1}(R)$ is defined locally by $y^2 = \sigma(x_1, \ldots, x_n) = 0$ while $R$ is defined locally as $\sigma(x_1, \ldots, x_n) = 0$. Also when $x \notin R$, $\sigma(x) \neq 0$ and thus we’ll have two solutions for $y^2 = \sigma(x)$, i.e. the fiber has two points.

It remains to show that the ramification locus is $R$. We recall what is the ramification locus defined in homework 1. It’s defined to be the place where $d\phi$ is not surjective.

As a map between $\phi : K \to L$, we have $d\phi(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_j} g$, and $d\phi(\frac{\partial}{\partial \tau}) = 2\tau \frac{\partial}{\partial \tau}$.

Now we restrict $Y$ to the locus $f^{-1}(0)$ and restrict $\Sigma$ to $g^{-1}(0)$ and calculate there.

We first look at the basis for holomorphic tangent bundle of $g^{-1}(0)$. Since

$$\left(\frac{\partial}{\partial x_l} - \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial \rho}\right) g = 0,$$

we know that $(\frac{\partial}{\partial x_l} - \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial \rho})_{1 \leq l \leq n}$ is a spanning set for $T \Sigma$ locally.

For $TY$, if $\sigma(z_1, \ldots, z_n) \neq 0$ locally near $p = (z_1, \ldots, z_n, \tau_0)$, we could find a local chart where $\tau \neq 0$ and use the spanning set $\{2\tau \frac{\partial}{\partial x_l} - \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial \tau}\}_{1 \leq l \leq n}$. Note that

$$d\phi \left(2\tau \frac{\partial}{\partial x_l} - \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial \tau}\right) = 2\tau \left(\frac{\partial}{\partial x_l} - \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial \rho}\right)$$

So when $\tau$ is nonzero, this map is an isomorphism between $n-$vector spaces.

If $\sigma(z_1, \ldots, z_n) = 0$, and since $R$ is not singular, we could find locally an $m$ such that $\frac{\partial \sigma}{\partial x_m}$ is non-vanishing, thus we have the spanning set $\{\frac{\partial}{\partial x_l} + \frac{\partial \sigma}{\partial x_l} \frac{\partial}{\partial x_m}\}_{l \neq m}$ as well as $\frac{\partial}{\partial \tau} - \frac{\partial \sigma}{\partial x_m} \frac{\partial}{\partial x_m}$.

But now

$$d\phi \left(\frac{\partial}{\partial \tau} - \frac{2\tau}{\partial \sigma/\partial x_m} \frac{\partial}{\partial x_m}\right) = 2\tau \left(\frac{\partial}{\partial \rho} + \frac{1}{\partial \sigma/\partial x_m} \frac{\partial}{\partial x_m}\right) = 0$$

when $\tau = 0$.

This happens exactly when $\sigma(z_1, \ldots, z_n) = 0$, and so $d\phi$ isn’t an isomorphism exactly when $\sigma(x_1, \ldots, x_m) = 0$, i.e. $x \in R$.

2. Let $X$ be a compact complex curve and let $f$ be a non-constant meromorphic function on $X$.

(a) Show that we can view $f$ as a holomorphic map from $X$ to $\mathbb{P}^1$.

We define $\tilde{f} : X \to \mathbb{P}^1$ to be $\tilde{f}(x) = [f(x) : 1]$ if $f(x) \in \mathbb{C}$ and $\tilde{f}(x) = [1 : 0]$ if $x$ is a pole of $f$.

Now we need to show that it’s a holomorphic map to $\mathbb{P}^1$.
For $x$ such that $f(x) \in \mathbb{C}$, of course locally it’s a holomorphic map to $\mathbb{P}^1$, since in the chart $\mathbb{C} \subset \mathbb{P}^1$, the function and the map to $\mathbb{P}^1$ have identical descriptions. So it suffices to show that when $x$ is a pole of $f$, then $f(z)$ is a holomorphic map near $x$.

In the chart of $\mathbb{P}^1$ at $[1,0]$, the mapping is described by the function $1/f(z)$, so it suffices to prove that $1/f(z)$ is locally a holomorphic function. In a local coordinate $z$ on $X$ centered at $x$ (which means that $x$ corresponds to $z = 0$), we can write $f(z) = z^{-k}h(z)$ for some $k \geq 1$ and holomorphic function $h(z)$ with $h(0) \neq 0$. Then $1/f(z) = z^k/h(z)$. We can shrink the coordinate neighborhood of $x$ if necessary that $h(z)$ has no zeros. Then $1/h(z)$ is holomorphic, hence $f(z)$ is a holomorphic map near $x$ as well.

(b) Let $t$ be a point of $\mathbb{P}^1$ ($t \neq \infty$) and let $D$ be a disk in $\mathbb{P}^1$ centered at $t$. Using exercise 3(c) of chapter 1, show that

$$n_t = \int_{f^{-1}(\partial D)} \frac{1}{2\pi i} \frac{df}{f-t}$$

For each point $t$ and we let $f^{-1}(t) = \{x_i\mid i \in I\}$. We first claim that $I$ is a finite set. This is because if it’s infinite and since $X$ is compact, we could find an accumulation point and by the same argument in homework 1, problem 3(a), we know that this is not possible since it would imply that $f(x) = t$ is a constant function.

Now we see that at each preimage $x_i$, there’s a neighborhood $U_i$ of $x_i$, so that $f(z) - t = \sum_{k \geq k_i} a_k(z-x_i)^k$, $\forall z \in U_i$, so that $f(z) - t = (z-x_i)^k \cdot \phi(z)$, where here $\phi(z)$ is a non-zero holomorphic function. Thus locally we can choose a change of variable $w = (z-x_i) \sqrt[k]{\phi(z)}$ by choosing one branch of $\sqrt[k]{\phi(z)}$. The Jacobian of this transformation is

$$\frac{\partial w}{\partial z} = k\sqrt[k]{\phi(z)} + (z-x_i) \frac{\phi'(z)}{(\sqrt[k]{\phi(z)})^{k_i-1}}.$$

So by shrinking the $U_i$ if necessary, we can find a non-zero Jacobian. So we switch to this local chart on $X$, with local coordinate $w$ and so $f(w) - t = w^k$ on this local chart.

Since $f(X - \bigcup_{i \in I} U_i)$ is compact as $f$ is continuous, we can find a $D'$ such that $\overline{D'} \cap f(X - \bigcup_{i \in I} U_i) = \emptyset$. So $f^{-1}(\partial D') \subset \bigcup_{i \in I} U_i$. Thus on $D'$ one can show that

$$f^{-1}(\partial D') = \bigcup_{i \in I} f^{-1}(\partial D') \cap U_i.$$

As a result, we can get for $D'$ the integral

$$n_t = \int_{f^{-1}(\partial D')} \frac{df}{f-t} = \sum_{i \in I} \int_{f^{-1}(\partial D') \cap U_i} \frac{dw^{k_i}}{w^{k_i}} = (2\pi i) \sum_{i \in I} k_i$$

We know as well that if we replace $D'$ by a smaller subset $D''$, the above argument still holds since $D'' \cap f(X - \bigcup_{i \in I} U_i) = \emptyset$ as well.

Now we have to prove that the integral does not depends on the choice of $D$. Without loss of generality we can assume that $D' \subset D$, this is because since $D' \cap D \neq \emptyset$ we can replace our $D'$ by $D' \cap D$ and the previous argument still holds. Thus we have to show that

$$\int_{f^{-1}(\partial D)} - \int_{f^{-1}(\partial D')} \frac{df}{f-t} = \int_{f^{-1}(\partial(D-D'))} \frac{df}{f-t} = 0.$$

We have $f^{-1}(\partial(D-D')) = \partial f^{-1}(D-D')$. Since $\frac{df}{f-t}$ is a holomorphic form on $f^{-1}(D-D')$, we can conclude from Stokes theorem that

$$\int_{\partial f^{-1}(D-D')} \frac{df}{f-t} = \int_{f^{-1}(D-D')} \partial (\frac{df}{f-t}) = 0.$$
On the other hand, if \( t = \infty \), we can do the same process for \( \frac{1}{f(z)} = \sum_{k \geq k_i} a_k(z - x_i)^k \), and now we can choose a local chart so that \( \frac{1}{f} = w^{k_i} \), and we can show that \( n_\infty \) is defined to be

\[
n_\infty = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{dg}{g} \quad \text{here } g = \frac{1}{f}.
\]

To prove that there’s an actual correspondence between this definition and our previous definition when \( t \neq \infty \), we can define

\[
m_s = \int_{\partial \Omega} \frac{dg}{g - s}, \quad \text{here } g = \frac{1}{f}.
\]

Now we need to prove that \( m_s \) actually parametrize \([1 : s]\) at the neighborhood of \( \infty = [1 : 0] \). By transformation of coordinate \([t : 1] = [1 : 1/t] \), thus we just need to show that \( m_{1/t} = n_t \) and so \( n_t \) naturally extends to \( t = \infty \).

To show this, we notice that if \( f - t \) has zero of order \( k_i \), \( 1/f - 1/t \) will have zero of order \( k_i \) as well. We can see this by the Taylor expansion \( 1/f - 1/t = (t - f)/(tf) \) and \( f - t \) has zero of order \( k_i \). Now integrating \( d(1/f)/(1/f - 1/t) \) at the boundary will give rises to the same integer at \( x_i \) since both of \( f - t \) and \( 1/f - 1/t \) has zero of order \( k_i \).

(c) Show that \( n_t \) is independent of \( t \).

We first claim that \( n_t \) defines a continuous function on both charts of \( \mathbb{P}^1 \). This is because when we fix \( D, \forall t, t' \in D \) in the chart \( U_1 \) not containing \([1 : 0]\) we’re integrating over the same path and thus the change is only

\[
\int_{f^{-1}(\partial \Omega)} (t - t') \frac{df}{(f - t)(f - t')}.
\]

Thus when \( t \to t' \) we see that the limit will be zero since \( \int \frac{df}{(f - t)^2} \) is finite.

Notice now if on the neighborhood of \( t = \infty \), as we have seen before, the integral is the same as

\[
m_s = \int_{g^{-1}(\partial \Omega)} \frac{dg}{g - s}, \quad g = \frac{1}{f}
\]

We know that \( m_s \) is continuous at neighborhood \( U_2 \) of \( s = 0 \) and we have the identity \( n_t = m_{1/t} \) when \( t \neq 0, \infty \).

Thus \( n_t : U_1 \to \mathbb{Z} \) is a continuous function, and the only possibility is that \( n_t \) is a constant on \( U_1 \). Since \( m_s \) is continuous on \( U_2 \), this tells us \( n_\infty = m_0 \) is the same constant as \( n_t \).

(d) Show that \( f \) is ramified at a point \( x \) if and only if the order of vanishing \( k_x(f - t) \) of \( f - t \) of \( x \) is at least equal to \( 2 \). Deduce from Sard’s theorem that for \( t \) in a dense set of points, the fiber \( f^{-1}(t) \) is a set of finite cardinality \( n \).

Fact: let \( g(z) = z^n \), then if the coordinate of the image is \( w \), then \( dg(\frac{\partial}{\partial z}) = nz^{n-1} \frac{\partial}{\partial w} \). So \( g \) is ramified at 0 if and only if \( n \geq 2 \).

Now if \( t \neq \infty \), by previous argument we see that we can find a neighborhood \( U_i \) of \( x \) an invertible map \( w^{-1} : U_i \to V_i \) such that \( f : V_i \to \mathbb{C} \) is given by \( w^{k_i} \). Since the Jacobian of \( w^{-1} \) is nonzero, we know that \( f \) is ramified at \( x \in U_i \) if and only if the induced map \( \tilde{f} \) is ramified at 0. We already know that \( z^k \) is ramified at 0 if and only if \( k \geq 2 \), thus the first claim is proven.

Sard’s theorem asserts that the places where \( df_x \) has rank less than 1 is of measure 0. Thus for a dense open set \( df_x \) is unramified, thus \( k_i = 1 \) for each preimage. Now since the degree is \( n \) we should have \( n \) points as our preimage.

If \( t = \infty \), this follows from the same argument with the function \( 1/f \).

(e) Deduce from (c) that the divisor \( f^{-1}(0) - f^{-1}(\infty) \) is of degree 0.

This follows from the fact that \( \deg f^{-1}(t) = \sum k_{x_i} = n_t \). Thus \( \deg f^{-1}(0) - f^{-1}(\infty) = n_0 - n_\infty = 0 \) since \( n_t \) is a constant function.
3. Show using the maximum principle that a connected compact complex manifold $X$ possesses no holomorphic functions other than the constant ones.

Let $|f| : X \to \mathbb{R}$ be the modulus of the holomorphic function $f$ and let $c$ be the maximum of $|f|$, which exists since $X$ is compact and $|f|$ is continuous. Now we set $U = \{z | |f(z)| = c\}$. Then $U$ is open by maximum principle (Theorem 1.21). Furthermore, $U$ is closed since $|f|$ is continuous. Since $U$ is nonempty as there exists a point achieving the maximum, we conclude that $U = X$.

4. Let $M = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the torus with its complex manifold structure. Prove that the holomorphic tangent bundle of $M$ is trivial.

The local charts of $M$ may be taken to be the open sets $\pi(B_p(1/2))$, where $B_p(1/2)$ is the disk of radius $1/2$ in $\mathbb{C}$ centered at $p$ and $\pi : \mathbb{C} \to M$ is the projection map.

Letting $z_j$ be the coordinate in one of these charts, then $TM$ is locally trivialized by $\partial/\partial z_j$. In comparing to another local coordinate $z_k$, we have $z_j = z_k + a_{jk} + ib_{ij}$ for some $a_{ij}, b_{ij} \in \mathbb{Z}$. It follows that $\partial/\partial z_j = \partial/\partial z_k$ and the transition functions for the tangent bundle are $\phi_{jk} \equiv 1$. As these are the transition functions of the trivial bundle bundle of rank 1, we conclude that $TM$ is trivial.