ON THE ABSENCE OF MCSHANE-TYPE IDENTITIES FOR THE OUTER SPACE

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Abstract. A remarkable result of McShane states that for a punctured torus with a complete finite volume hyperbolic metric we have

$$\sum_{\gamma} \frac{1}{e^{\ell(\gamma)}+1} = \frac{1}{2}$$

where γ varies over the homotopy classes of essential simple closed curves and $\ell(\gamma)$ is the length of the geodesic representative of γ.

We prove that there is no reasonable analogue of McShane's identity for the Culler-Vogtmann outer space of a free group.

1. Introduction

Let T be the one-punctured torus and let ρ be a complete finite-volume hyperbolic structure on T. Let $\mathcal{S}$ be the set of all free homotopy classes of essential simple closed curves in T that are not homotopic to the puncture. Denote

$$E(\rho) := \sum_{\gamma\in\mathcal{S}} \frac{1}{e^{\ell_\rho(\gamma)}+1},$$

where $\ell_\rho(\gamma)$ is the smallest ρ-length among all curves representing γ. Thus E can be regarded as a function on the Teichmüller space of T. A remarkable result of McShane [9] shows that this function is constant and that

$$(*) \qquad E(\rho) = \frac{1}{2}$$

for every ρ. We refer to $(*)$ as *McShane's identity* for T. Since the thesis of McShane [9], other proofs of McShane's identity for the punctured torus have been produced (particularly, see the work of Bowditch [3]) and McShane's identity has been generalized to other hyperbolic surfaces and other contexts [4, 10, 1, 2, 13, 14]. Note that if ψ is an element of the mapping class group of T then ψ permutes the elements of $\mathcal{S}$ and hence, clearly, $E(\rho) = E(\psi\rho)$. Thus E obviously factors through to a function on the moduli space of T and $(*)$ says that this function is identically equal to 1/2.

Let $F_k = F(a_1, \dots, a_k)$ be a free group of rank $k \geq 2$ with a free basis $A = \{a_1 \dots, a_k\}$. For F_k the best analogue of the Teichmüller space is the so-called Culler-Vogtmann *outer space* $CV(F_k)$. Instead of actions on the hyperbolic plane the elements of the outer space are represented by minimal discrete isometric actions of F_k on $\mathbb{R}$-trees. Equivalently, one can think about a point of the outer space as being represented by a *marked volume-one metric graph structure* on F_k, that is,

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an isomorphism $\phi : F_k \to \pi_1(\Gamma, p)$, where Γ is a finite graph without degree-one and degree-two vertices, equipped with a *metric structure* $\mathcal{L}$ that assigns to each non-oriented edge of Γ a positive number called the *length* of this edge. The *volume* of a metric structure on Γ is the sum of the lengths of all non-oriented edges of Γ. As we noted, the metric structures that appear in the description of the points of the outer space, given above, are required to have volume equal to one. If $(\phi : F_k \to \pi_1(\Gamma, p), \mathcal{L})$ represents a point of the outer space, the metric structure $\mathcal{L}$ naturally lifts to the universal cover $\widetilde{\Gamma}$, turning $\widetilde{\Gamma}$ into an $\mathbb{R}$-tree X. The group F_k acts on this $\mathbb{R}$-tree X via ϕ by isometries minimally and discretely with the quotient being equal to Γ. Similarly to the marked length spectrum in the Teichmüller space context, a marked metric graph structure $(\phi : F_k \to \pi_1(\Gamma, p), \mathcal{L})$ defines a *hyperbolic length function* $\ell : \mathcal{C}_k \to \mathbb{R}$ where $\mathcal{C}_k$ is the set of all nontrivial conjugacy classes in F_k. If $g \in F_k$, then $\ell([g])$ is the translation length of g considered as the isometry of the $\mathbb{R}$-tree X described above. Alternatively, we can think about $\ell([g])$ as follows: $\ell([g])$ is the $\mathcal{L}$-length of the shortest free homotopy representative of the curve $\phi(g)$ in Γ, that is, the $\mathcal{L}$-length of the "cyclically reduced" form of $\phi(g)$ in Γ. Two volume-one metric graph structures on F_k represent the same point of $CV(F_k)$ if and only if their corresponding hyperbolic length functions are equal, or, equivalently, if the corresponding $\mathbb{R}$-trees are F_k-equivariantly isometric.

It is natural to ask if there is an analogue of McShane's identity in the outer space context. The (right) action of $\psi \in Out(F_k)$ on $CV(F_k)$ takes a hyperbolic length function ℓ to $\ell \circ \psi$, that is, ψ simply permutes the domain $\mathcal{C}_k$ of ℓ. Therefore the real question, as in the Teichmüller space case, is if there is an analogue of McShane's identity for the *moduli space* $\mathcal{M}_k = CV(F_k)/Out(F_k)$. The elements of $\mathcal{M}_k$ are represented by unmarked finite connected volume-one metric graphs $(\Gamma, \mathcal{L})$ without degree-one and degree-two vertices and with $\pi_1(\Gamma) \simeq F_k$.

To simplify the picture, and also since our results will be negative, we will consider a subset Δ_k of $CV(F_k)$ consisting of all volume-one metric structures on the wedge W_k of k circles wedged at a base-vertex v_0. We orient the circles and label them by $a_1, \dots, a_k$. This gives us an identification $\pi_1(W_k, v_0) = F(a_1, \dots, a_k)$ of $F_k = F(a_1, \dots, a_k)$ with $\pi_1(W_k, v_0)$, so that indeed $\Delta_k \subseteq CV(F_k)$.

A volume-one metric structure $\mathcal{L}$ on W_k is a k-tuple $(\mathcal{L}(a_1), \dots, \mathcal{L}(a_k))$ of positive numbers with $\sum_{i=1}^k \mathcal{L}(a_i) = 1$. Thus Δ_k has the natural structure of an open $(k-1)$-dimensional simplex in $\mathbb{R}^k$. As in the general outer space context, every $\mathcal{L} \in \Delta_k$ defines a hyperbolic length-function $\ell_{\mathcal{L}} : \mathcal{C}_k \to \mathbb{R}$, where for $g \in F_k$ $\ell_{\mathcal{L}}([g])$ is the $\mathcal{L}$-length of the cyclically reduced form of g in $F_k = F(a_1, \dots, a_k)$. The open simplex Δ_k has a distinguished point $\mathcal{L}_* := (\frac{1}{k}, \dots, \frac{1}{k})$. Note that for every $[g] \in \mathcal{C}_k$ we have $\ell_{\mathcal{L}_*}([g]) = ||g||/k$, where $||g||$ is the cyclically reduced length of g in $F_k = F(a_1, \dots, a_k)$.

There is no perfect analogue for the notion of a simple closed curve in the free group context. The closest such analogue is given by *primitive elements*, that is, elements belonging to some free basis of F_k. Let $\mathcal{P}_k$ be denote the set of conjugacy classes of primitive elements of F_k. We will consider two versions of possible generalizations of McShane's identity for free groups: the first involving all conjugacy classes in F_k and the second involving the conjugacy classes of primitive elements of F_k. We will see that, under some reasonable assumptions, there are no analogues of McShane's identity in either context.

Definition 1.1 (McShane-type functions on Δ_k)**.** Let $f : (0,\infty) \to (0,\infty)$ be a monotone non-increasing function and let $k \geq 2$. Define

$$C_f : \Delta_k \to (0,\infty], \qquad C_f(\mathcal{L}) = \sum_{w \in \mathcal{C}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k$$

and

$$P_f : \Delta_k \to (0,\infty], \qquad P_f(\mathcal{L}) = \sum_{w \in \mathcal{P}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k.$$

Obviously, $0 < P_f < C_f \leq \infty$ on Δ_k.

Motivated by McShane's identity, it is interesting to ask if there exist functions f such that either C_f or P_f is constant on Δ_k. To make the question meaningful we need to require P_f (or, correspondingly, C_f) be finite at some point $\mathcal{L} \in \Delta_k$. Thus it is necessary to assume that $\lim_{x\to\infty} f(x) = 0$ and that this convergence to zero is sufficiently fast.

We establish the following negative results regarding the existence of analogues of McShane's identity in the outer space context:

Theorem A. *Let $k \geq 2$ be an integer and let $F = F(a_1, \dots, a_k)$. Let $f : (0,\infty) \to (0,\infty)$ be a monotone non-increasing function such that:*

(1)
$$\limsup_{x\to\infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

(2)
$$\liminf_{x\to\infty} f(x)^{1/x} > 0.$$

Then:

(a) *We have $P_f \leq C_g < \infty$ on some neighborhood U on $\mathcal{L}_*$ in Δ_k (moreover, only the assumption (1) on f above is required for this conclusion).*
(b) *We have $C_f \neq const$ on Δ_k.*
(c) *If $k \geq 3$ then $P_f \neq const$ on Δ_k.*

The assumptions on $f(x)$ in Theorem A require $f(x)$ to decay both at least and at most exponentially fast; condition (1) assures that the value of C_f is finite near $\mathcal{L}_*$. The idea of the proof of parts (b) and (c) of Theorem A uses the notion of *volume entropy* for a metric structure $\mathcal{L}$ on W_k (see [6, 11, 8]). Roughly speaking, there are points $\mathcal{L}$ near the the boundary of Δ_k where the exponential growth rate, as $R \to \infty$, of the number of conjugacy classes with $\ell_{\mathcal{L}}$-length at most R is bigger than the exponential rate of decay of the function f. This forces C_f to be equal to ∞ at $\mathcal{L}$.

For $k = 2$ the set of conjugacy classes of primitive elements has quadratic rather than exponential growth. Therefore we modify the assumptions on $f(x)$ accordingly and obtain a somewhat stronger conclusion then in part (c) of Theorem A. For $k = 2$ the open 1-dimensional simplex $\Delta_2 \subseteq \mathbb{R}^2$ consists of all pairs $\mathcal{L}_t := (t, 1-t)$ where $t \in (0,1)$. Therefore we may identify Δ_2 with $(0,1)$ and define $P_f(t) := P_f(\mathcal{L}_t)$. With this convention we prove:

Theorem B. *Let $k = 2$ and $F = F(a,b)$. Let $f : (0,\infty) \to (0,\infty)$ be a monotone non-increasing function such that:*

(1) *We have $f''(x) > 0$ for every $x > 0$.*
(2) *There is some $\epsilon > 0$ such that $\lim_{x\to\infty} x^{3+\epsilon} f(x) = 0$.*

Then the following hold:

(a) *We have* $0 < P_f(t) < \infty$ *for every* $t \in (0,1)$.
(b) *The function* $P_f(t)$ *is strictly convex on* $(0,1)$ *and achieves a unique minimum at* $t_0 = 1/2$. *In particular,* $P_f(t)$ *is not a constant locally near* $t_0 = 1/2$ *and thus* $P_f \neq const$ *on* $(0,1)$.

The proof of Theorem B uses convexity considerations as well as some results about the explicit structure of primitive elements in $F(a,b)$ [5, 12].

Finally, we combine the volume entropy and the convexity ideas to obtain:

Theorem C. *Let* $k \geq 2$ *and let* $f : (0,\infty) \to (0,\infty)$ *be a monotone decreasing function such that the following hold:*

(1) *The function* $f(x)$ *is strictly convex on* $(0,\infty)$.
(2)
$$\limsup_{x\to\infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

Then there exists a convex neighborhood U *of* $\mathcal{L}_*$ *in* Δ_k *such that* $0 < P_f < C_f < \infty$ *on* U *and both* C_f *and* P_f *are strictly convex on* U. *In particular,* $C_f \neq const$ *on* U *and* $P_f \neq const$ *on* U.

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2. Volume entropy

In this section we will prove Theorem A, which is obtained as a combination of Theorem 2.4 and Theorem 2.5 below.

Convention 2.1. For the remainder of this section let $k \geq 2$ be an integer and $F_k = F(a_1,\dots,a_k)$ be free of rank k with a free basis $A = \{a_1,\dots,a_k\}$. We identify F_k with $\pi_1(W_k, v_0)$, as explained in the introduction. For $g \in F_k$ we denote by $|g|$ the freely reduced length of g with respect to A and we denote by $||g||$ the cyclically reduced length of g with respect to A.

We denote by CR_k the set of all cyclically reduced elements of F_k with respect to A.

Let $\mathcal{L}$ be a metric graph structure on W_k. For every $g \in F_k$ there is a unique edge-path in W_k labelled by the freely reduced form of g with respect to A. We denote the $\mathcal{L}$-length of that path by $\mathcal{L}(g)$. As before, we denote by $\ell_{\mathcal{L}} : \mathcal{C}_k \to \mathbb{R}$ the hyperbolic length function corresponding to $\mathcal{L}$. Thus if $g \in F_k$ then $\ell_{\mathcal{L}}([g]) = \mathcal{L}(u)$ where u is the cyclically reduced form of g with respect to A.

Definition 2.2 (Volume Entropy)**.** Let $\mathcal{L}$ be a metric structure on W_k. The *volume entropy* $h_{\mathcal{L}}$ of $\mathcal{L}$ is defined as
$$h_{\mathcal{L}} = \lim_{R\to\infty} \frac{\log \#\{g \in F_k : \mathcal{L}(g) \leq R\}}{R}.$$

It is well-known and easy to see that the limit in the above expression exists and is finite. We refer the reader to [6, 11, 8] for a detailed discussion of volume entropy in the context of metric graphs.

Proposition 2.3. *Let* $k \geq 2$ *and* $\mathcal{L}$ *be as in definition 2.2.*

Then the limits
$$h'_{\mathcal{L}} = \lim_{R\to\infty} \frac{\log \#\{g \in CR_k : \mathcal{L}(g) \leq R\}}{R}.$$

and

$$h''_{\mathcal{L}} = \lim_{R\to\infty} \frac{\log \#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \le R\}}{R}.$$

exist and

$$h_{\mathcal{L}} = h'_{\mathcal{L}} = h''_{\mathcal{L}}.$$

Proof. Let $M := \max\{|a_i|_{\mathcal{L}} : i = 1, \dots, k\}$ and $m := \min\{|a_i|_{\mathcal{L}} : i = 1, \dots, k\}$

For each $g \in F$ there exists a cyclically reduced word v_g such that $|g| = |v_g|$ and such that g and v_g agree except possibly in the last letter. Then $\big|\mathcal{L}(g) - \mathcal{L}(v_g)\big| \le M$. Moreover, the function $F_k \to CR_k, g \mapsto v_g$ is at most $2k$-to-one. Therefore for every integer $R > 0$

$$\#\{g \in CR_k : \mathcal{L}(g) \le R\} \le \#\{g \in F_k : \mathcal{L}(g) \le R\} \le 2k\#\{g \in CR_k : \mathcal{L}(g) \le R+M\}$$

and

$$\#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \le R\} \le \#\{g \in CR_k : \mathcal{L}(g) \le R\} \le \frac{R}{m}\#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \le R\}.$$

This implies the statement of the proposition. □

Theorem 2.4. *Let $k \ge 2$ be an integer and let $F = F(a_1, \dots, a_k)$. Let $f : (0, \infty) \to (0, \infty)$ be a monotone non-increasing function such that:*

(1)
$$\limsup_{x\to\infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

(2)
$$\liminf_{x\to\infty} f(x)^{1/x} > 0.$$

Then:

(a) *We have $0 < C_f < \infty$ on some neighborhood U on $\mathcal{L}_*$ in Δ_k (moreover, only the assumption (1) on f above is required for this conclusion).*

(b) *We have $C_f \ne const$ on Δ_k.*

Proof. The assumptions on $f(x)$ imply that there exist $N > 0$ and $0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^k}$ such that for every $x \ge N$

$$\sigma_1^x \le f(x) \le \sigma_2^x.$$

Let $\mathcal{L}_* = (\frac{1}{k}, \dots, \frac{1}{k}) \in \Delta_k$. For any $g \in F_k$ we have $\mathcal{L}_*(g) = |g|/k$. Then an easy direct computation shows that $h_{\mathcal{L}_*} = k\log(2k-1)$, so that $e^{h_{\mathcal{L}_*}} = (2k-1)^k < \frac{1}{\sigma_2}$. Since the volume entropy h is a continuous function on Δ_k (see, for example, [6]), there exist a neighborhood U of $\mathcal{L}_*$ in Δ_k and $0 < c < \frac{1}{\sigma_2}$ such that $e^{h_{\mathcal{L}}} < c$ for every $\mathcal{L} \in U$.

Observe now that $C_f < \infty$ on U. Let $\mathcal{L} \in U$ be arbitrary. There there exist $M > 0$ and c_1 with $c < c_1 < \frac{1}{\sigma_2}$ such that for every integer $R > 0$ we have

$$\#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \le R\} \le Mc_1^R$$

Therefore

$$C_f(\mathcal{L}_*) = \sum_{w\in\mathcal{C}_k} f(\ell_{\mathcal{L}}(w)) = \sum_{i=0}^{\infty} \sum_{w\in\mathcal{C}_k, i<\ell_{\mathcal{L}}(w)\le i+1} f(\ell_{\mathcal{L}}(w)) \le$$

$$\sum_{i=0}^{\infty} \sum_{w\in\mathcal{C}_k, i<\ell_{\mathcal{L}}(w)\le i+1} f(i) \le \sum_{i=0}^{\infty} M c_1^{i+1} f(i) < \infty$$

where the last inequality holds since $c_1 < \frac{1}{\sigma_2}$ and $f(x) \le \sigma_2^x$ for all $x \ge N$. Thus indeed $C_f(\mathcal{L}) < \infty$, so that $C_f < \infty$ on U.

For $0 < t < \frac{1}{k-1}$ let $\mathcal{L}_t = (t, t, \dots, t, 1-(k-1)t) \in \Delta_k$. Then, as follows from the proof of Theorem B of [6] (specifically the proof of Theorem 9.4 on page 25 of [6]),

$$\lim_{t\to 0} h_{\mathcal{L}_t} = \infty.$$

Indeed, let Γ be the subgraph of W_k consisting of the loops labelled by a_1 and a_k. The restriction $\mathcal{L}_t'$ of $\mathcal{L}_t$ to Γ is a metric structure on Γ of volume $1-(k-2)t$. Therefore $\frac{1}{1-(k-2)t}\mathcal{L}_t'$ is a volume-one metric structure on Γ with respect to which the length of a_1 goes to 0 as $t \to 0$. Therefore, as established in the proof of Theorem 9.4 of [6],

$$\lim_{t\to 0} h_{\frac{1}{1-(k-2)t}\mathcal{L}_t'} = \infty.$$

However,

$$h_{\mathcal{L}_t'} = \frac{1}{1-(k-2)t} h_{\frac{1}{1-(k-2)t}\mathcal{L}_t'}$$

and therefore

$$\lim_{t\to 0} h_{\mathcal{L}_t'} = \infty.$$

It is obvious from the definition of volume entropy that $h_{\mathcal{L}_t'} \le h_{\mathcal{L}_t}$ and hence

$$\lim_{t\to 0} h_{\mathcal{L}_t} = \infty,$$

as claimed.

Hence there exists $t_0 \in (0, \frac{1}{k-1})$ such that for every $t \in (0, t_0)$ we have $e^{h_{\mathcal{L}_t}} > \frac{1}{\sigma_1} + 2$. Let $t \in (0, t_0)$ be arbitrary. We claim that $C_f(\mathcal{L}_t) = \infty$.

Since $e^{h_{\mathcal{L}_t}} > \frac{1}{\sigma_1} + 2$, by Proposition 2.3 there is $R_0 > N > 0$ such that for every $R \ge R_0$ we have

$$\#\{w \in \mathcal{C}_k : \mathcal{L}_t(w) \le R\} \ge (\frac{1}{\sigma_1} + 1)^R.$$

For every $R \ge R_0$

$$C_f(\mathcal{L}_t) = \sum_{w\in\mathcal{C}_k} f(\mathcal{L}_t(w)) \ge \sum_{w\in\mathcal{C}_k, \mathcal{L}_t(w)\le R} f(\mathcal{L}_t(w)) \ge$$

$$\sum_{w\in\mathcal{C}_k, \mathcal{L}_t(w)\le R} f(R) \ge (\frac{1}{\sigma_1}+1)^R f(R) \ge (\frac{1}{\sigma_1}+1)^R \sigma_1^R =$$

$$= (1+\sigma_1)^R.$$

Since this is true for every $R \ge R_0$, it follows that $C_f(\mathcal{L}_t) = \infty$.

Thus $C_f(\mathcal{L}_*) < \infty$ while $C_f(\mathcal{L}_t) = \infty$ for all sufficiently small $t > 0$. Therefore $C_f \ne const$ on Δ_k. □

Theorem 2.5. *Let $k \ge 3$ be an integer and let $F = F(a_1, \dots, a_k)$. Let $f : (0, \infty) \to (0, \infty)$ be as in Theorem 2.4.*

Then:

(a) *We have $P_f \le C_f < \infty$ on some neighborhood U on $\mathcal{L}_*$ in Δ_k.*
(b) *We have $P_f \neq const$ on Δ_k.*

Proof. Again, by assumptions on $f(x)$, there exist there exist $N > 0$ and $0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^k}$ such that for every $x \ge N$

$$\sigma_1^x \le f(x) \le \sigma_2^x.$$

By Definition $0 \le P_f \le C_f$. By Theorem 2.4 we have $C_f < \infty$ on some neighborhood U on $\mathcal{L}_*$ in Δ_k and hence $P_f \le C_f < \infty$ on U.

Put $F_{k-1} := F(a_1, \dots, a_{k-1})$ so that $F_k = F_{k-1} * \langle a_k \rangle$. For $0 < t < \frac{1}{k-2}$ let

$$\mathcal{L}_t := (\frac{t}{2}, \frac{t}{2}, \dots, \frac{t}{2}, \frac{1}{2} - (k-2)\frac{t}{2}, \frac{1}{2}) \in \Delta_k$$

and

$$\widehat{\mathcal{L}_t} := (\frac{t}{2}, \frac{t}{2}, \dots, \frac{t}{2}, \frac{1}{2} - (k-2)\frac{t}{2}) \in \frac{1}{2}\Delta_{k-1}.$$

Thus $2\widehat{\mathcal{L}_t} \in \Delta_{k-1}$ is a volume-one metric structure on W_{k-1}. Since $k \ge 3$, we have $k - 1 \ge 2$ and hence, exactly as in the proof of Theorem 2.4, $\lim_{t \to 0} h_{2\widehat{\mathcal{L}_t}} = \infty$.

For $R \ge 1$

$$b_{R,t} := \#\{g \in F_{k-1} : \widehat{\mathcal{L}_t}(g) \le R\}.$$

Then

$$s_t := \lim_{R \to \infty} \frac{\log b_{R,t}}{R} = h_{\widehat{\mathcal{L}_t}} = 2h_{2\widehat{\mathcal{L}_t}}.$$

and therefore

$$\lim_{t \to 0} s_t = \infty.$$

Hence there exists $0 < t_0 < \frac{1}{k-2}$ such that for every $t \in (0, t_0)$ we have $e^{s_t} > \frac{1}{\sigma_1} + 2$.

Fix an arbitrary $t \in (0, t_0)$. Since $e^{s_t} > \frac{1}{\sigma_1} + 2$, by there is $R_0 > N > 0$ such that for every $R \ge R_0$ we have

$$b_{R,t} = \#\{g \in F_{k-1} : \widehat{\mathcal{L}_t}(g) \le R\} \ge (\frac{1}{\sigma_1} + 1)^R.$$

Note that for every $g \in F_{k-1}$ the element $g a_k \in F_k$ is primitive in F. Moreover, if $g_1 \neq g_2$ are distinct elements of F_{k-1} then $g_1 a_k$ and $g_2 a_k$ are not conjugate in F_k. Recall that by definition of $\mathcal{L}_t$ we have $\mathcal{L}_t(a_k) = \frac{1}{2}$. For $R \ge 1$ denote

$$p_{R,t} := \#\{w \in \mathcal{P}_k : \ell_{\mathcal{L}_t}(w) \le R\}.$$

Then for every $R \ge R_0 + \frac{1}{2}$ we have

$$p_{R,t} \ge b_{R-\frac{1}{2},t} \ge (\frac{1}{\sigma_1} + 1)^{R-\frac{1}{2}}.$$

Hence for every $R \ge R_0 + \frac{1}{2}$

$$P_f(\mathcal{L}_t) = \sum_{w \in \mathcal{P}_k} f(\mathcal{L}_t(w)) \ge \sum_{w \in \mathcal{P}_k, \ell_{\mathcal{L}_t}(w) \le R} f(\mathcal{L}_t(w)) \ge$$

$$\sum_{w \in \mathcal{P}_k, \ell_{\mathcal{L}_t}(w) \le R} f(R) \ge (\frac{1}{\sigma_1} + 1)^{R-\frac{1}{2}} f(R) \ge (\frac{1}{\sigma_1} + 1)^{R-\frac{1}{2}} \sigma_1^R =$$

$$= (1+\sigma_1)^R (\frac{1}{\sigma_1} + 1)^{-\frac{1}{2}}.$$

Since this is true for every $R \ge R_0 + \frac{1}{2}$, it follows that $P_f(\mathcal{L}_t) = \infty$.

Thus $P_f(\mathcal{L}_*) < \infty$ while $P_f(\mathcal{L}_t) = \infty$ for all sufficiently small $t > 0$. Therefore $P_f \ne const$ on Δ_k.

□

3. Primitive elements in $F(a, b)$

In this section we will prove Theorem B.

Convention 3.1. Throughout this section let $F_2 = F(a, b)$ be a free group of rank two.

Let $\alpha : F(a, b) \to \mathbb{Z}^2$ be the abelianization homomorphism, that is, $\alpha(a) = (1, 0)$ and $\alpha(b) = (0, 1)$. Then α is constant on every conjugacy class and therefore α defines a map $\beta : \mathcal{C}_2 \to \mathbb{Z}^2$.

Definition 3.2 (Visible points)**.** A point $(p, q) \in \mathbb{Z}^2$ is called *visible* if $gcd(p, q) = 1$. We denote the set of all visible points in $\mathbb{Z}^2$ by V.

We will need the following known facts about primitive elements in $F(a, b)$ (see, for example, [5, 12]:

Proposition 3.3. *The following hold:*

(1) *The restriction of β to $\mathcal{P}_2$ is a bijection between $\mathcal{P}_2$ and the set of visible elements $V \subseteq \mathbb{Z}^2$.*
(2) *Let $w \in F(a, b)$ be a cyclically reduced primitive element and let $\alpha(w) = (p, q) \in \mathbb{Z}^2$.*
   Then every occurrence of a in w has the same sign (either $-1, 0$ or 1) as p and every occurrence of b in w has the same sign (again either $-1, 0$ or 1) as q. Thus the total number of occurrences of $a^{\pm 1}$ in w is equal to $|p|$ and the total number of occurrences of $b^{\pm 1}$ in w is equal to $|q|$.

Definition 3.4 (Admissible function)**.** We say that a function $f : (0, \infty) \to [0, \infty)$ is *admissible* if it satisfies the following conditions:

(1) We have $f''(x) > 0$ for every $x > 0$.
(2) There is some $\epsilon > 0$ such that $\lim_{x \to \infty} x^{3+\epsilon} f(x) = 0$.

The second condition means that $f(x)$ converges to zero asymptotically faster than $\frac{1}{x^{3+\epsilon}}$ as $x \to \infty$. Note that an admissible function must be strictly positive and monotone decreasing on $(0, \infty)$.

Theorem 3.5. *Let f be any admissible function. Then the following hold:*

(1) *We have $0 < P_f(t) < \infty$ for every $t \in (0, 1)$.*
(2) *The function $P_f(t)$ is strictly convex on $(0, 1)$ and achieves a unique minimum at $t = 1/2$. In particular, $P_f(t)$ is not a constant locally near $t = 1/2$.*

Proof. For every $(p,q) \in V$ and $t \in (0,1)$ denote

$$g_{p,q}(t) = f(t|p| + (1-t)|q|) + f(t|q| + (1-t)|p|).$$

Note that if $(p,q) \in V$ then $gcd(p,q) = 1$ and hence $|p| \neq |q|$. We can therefore partition V as the collection of pairs $(p,q),(q,p)$ of visible elements and every such pair has a unique representative where the absolute value of the first coordinate is bigger than that of the second coordinate.

Let $w \in \mathcal{P}_2$ be arbitrary and let $(p,q) = \beta(w)$. Proposition 3.3 and the definition of $\mathcal{L}_t$ imply that for any $t \in (0,1)$ we have

$$\ell_{\mathcal{L}_t}(w) = t|p| + (1-t)|q|.$$

Let $V' := \{(p,q) \in V : |p| > |q|\}$. Then we have

$$(\dagger) \qquad P_f(t) = \sum_{w \in \mathcal{P}_2} f(\ell_{\mathcal{L}_t}(w)) = \sum_{(p,q)\in V} f(t|p| + (1-t)|q|) =$$

$$\sum_{(p,q)\in V'} f(t|p| + (1-t)|q|) + f(t|q| + (1-t)|p|) = \sum_{(p,q)\in V'} g_{p,q}(t).$$

Fix some $t \in (0,1)$. We can also represent $P_f(t)$ as

$$P_f(t) = \sum_{N=1}^{\infty} \sum_{(p,q)\in V, \max\{|p|,|q|\}=N} f(t|p| + (1-t)|q|).$$

Since $f(x)$ is a monotone non-increasing function, if $(p,q) \in V, \max\{|p|,|q|\} = N$, we have

$$f(t|p| + (1-t)|q|) \leq \min\{f(tN), f((1-t)N)\} = f(cN)$$

where $c = \max\{t, 1-t\}$. For every integer $N \geq 1$ the number of points $(p,q) \in \mathbb{Z}^2$ with $|p| \leq N, |q| \leq N$ is $(2N+1)^2$.

Therefore

$$P_f(t) = \sum_{N=1}^{\infty} \sum_{(p,q)\in V, \max\{|p|,|q|\}=N} f(t|p| + (1-t)|q|) \leq$$

$$\sum_{N=1}^{\infty} \sum_{(p,q)\in V, \max\{|p|,|q|\}=N} f(cN) \leq \sum_{N=1}^{\infty} (2N+1)^2 f(cN) < \infty$$

because of condition (2) in the definition of admissibility of $f(x)$. Thus $0 < P_f(t) < \infty$ for every $t \in (0,1)$.

Note that for each $(p,q) \in V'$

$$g'_{p,q}(t) = f'(t|p| + (1-t)|q|)(|p| - |q|) + f'(t|q| + (1-t)|p|)(|q| - |p|)$$

$$g''_{p,q}(t) = f''(t|p| + (1-t)|q|)(|p| - |q|)^2 + f''(t|q| + (1-t)|p|)(|q| - |p|)^2$$

Since $|p| > |q|$ and, by definition of admissibility, $f''(x) > 0$ for every $x \in \mathbb{R}$, we conclude that $g''_{p,q}(t) > 0$ for every $t \in (0,1)$. Hence $g_{p,q}$ is strictly convex on $(0,1)$. Moreover,

$$g'_{p,q}(\frac{1}{2}) = f'(\frac{|p|}{2} + \frac{|q|}{2})(|p| - |q|) + f'(\frac{|q|}{2} + \frac{|p|}{2})(|q| - |p|) = 0.$$

Since $g''_{p,q} > 0$ on $(0,1)$, it follows that $g_{p,q}$ is strictly convex on $(0,1)$ and achieves a unique minimum on $(0,1)$ at $t = \frac{1}{2}$.

Since $0 < P_f < \infty$ on $(0,1)$ and $P_f = \sum_{(p,q)\in V'} g_{p,q}$, it also follows that P_f is strictly convex on $(0,1)$ and achieves a unique minimum on $(0,1)$ at $t = \frac{1}{2}$. $\square$

4. Exploiting convexity

In this section we combine the ideas of the previous two sections and establish Theorem C from the introduction.

Theorem 4.1. *Let $k \geq 2$ and let $f : (0,\infty) \to (0,\infty)$ be monotone decreasing function such that the following hold:*

(1) *The function $f(x)$ is strictly convex on $(0,\infty)$.*
(2)
$$\limsup_{x\to\infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

Then there exists a convex neighborhood U of $\mathcal{L}_*$ in Δ_k such that $0 < P_f < C_f < \infty$ on U and both C_f and P_f are strictly convex on U. In particular, $C_f \neq const$ on U and $P_f \neq const$ on U.

Proof. By Theorem 2.4 there exists a convex neighborhood U of $\mathcal{L}_*$ in Δ_k, such that $0 < P_f < C_f < \infty$ on U. We will prove that P_f and C_f are strictly convex on U.

Let D be the set of all k-tuples of integers $m = (m_1, \dots, m_k)$ such that $m_i \geq 0$ for $i = 1, \dots, k$ and $m_1 + \dots + m_k > 0$. For each $m = (m_1, \dots, m_k) \in D$ let Q_m be the set of all $w \in \mathcal{C}_k$ such that w involves exactly m_i occurrences of $a_i^{\pm 1}$ for $i = 1, \dots, k$ and let $q_m := \#(Q_m)$. Note that for every $w \in Q_m$, if $\mathcal{L} = (x_1, \dots, x_k) \in \Delta_k$, then we have
$$\ell_{\mathcal{L}}(w) = m_1 x_1 + \dots m_k x_k.$$
Denote by $f_m : \Delta_k \to \mathbb{R}$ the function defined as
$$f_m(x_1, \dots, x_k) := f(m_1 x_1 + \dots + m_k x_k), \quad (x_1, \dots, x_k) \in \Delta_k.$$
The function $f(x)$ is convex on $(0,\infty)$ and the function $(x_1, \dots, x_k) \mapsto m_1 x_1 + \dots + m_k x_k$ is linear on Δ_k. Therefore f_m is convex on Δ_k.

Then for any $\mathcal{L} = (x_1, \dots, x_k) \in \Delta_k$ we have
$$C_f(\mathcal{L}) = \sum_{m\in D} q_m f_m(\mathcal{L}).$$
Since each f_m is convex on Δ_k, it follows that C_f is convex on Δ. We claim that C_f is strictly convex on U. Let D_1 be the subset of D consisting of all the k-tuples having a single nonzero entry equal to 1, that is, D_1 is the union of the k standard unit vectors in $\mathbb{Z}^k$. Let $m_i = (0, \dots, 1, \dots 0) \in D_1$ where 1 occurs in the i-th position. Then $Q_{m_i} = \{[a_i], [a_i^{-1}]\}$ and $q_{m_i} = 2$. Also, $f_{m_i}(x_1, \dots, x_k) = f(x_i)$ for every $(x_1, \dots, x_k) \in \Delta_k$.

Put $g := f_{m_1} + \dots + f_{m_k} : \Delta_k \to \mathbb{R}$, so that
$$g(x_1, \dots, x_k) = f(x_1) + \dots + f(x_k), \quad (x_1, \dots, x_k) \in \Delta_k.$$
It is easy to see that g is strictly convex on Δ_k since f is strictly convex on $(0,\infty)$. We have:
$$C_f = \sum_{m\in D} q_m f_m = 2g + \sum_{m\in D-D_1} q_m f_m$$

Since $C_f < \infty$ on a convex set U and since g is strictly convex on U and $\sum_{m\in D-D_1} q_m f_m$ is convex on U, it follows that C_f is strictly convex on U as claimed.

The proof that P_f is strictly convex on U is exactly the same as for C_f above. The only change that needs to be made is to re-define Q_m for each $m = (m_1, \dots, m_k) \in D$ as the set of all $w \in \mathcal{P}_k$ such that w involves exactly m_i occurrences of $a_i^{\pm 1}$ for $i = 1, \dots, k$. □

Remark 4.2. Let $\mathcal{Z}_k$ be the set of all *root-free* conjugacy classes $w \in \mathcal{C}_k$, that is, conjugacy classes of nontrivial elements of F_k that are not proper powers. It is not hard to show, similar to Proposition 2.3, that if $\mathcal{L}$ is a metric structure on W_k then

$$h_{\mathcal{L}} = \widetilde{h}_{\mathcal{L}}$$

where

$$\widetilde{h}_{\mathcal{L}} := \lim_{R\to\infty} \frac{\log \#\{w \in \mathcal{Z}_k : \ell_{\mathcal{L}}(w) \le R\}}{R}.$$

If one now re-defines the McShane function C_f as S_f:

$$S_f : \Delta_k \to (0, \infty], \qquad S_f(\mathcal{L}) = \sum_{w\in\mathcal{Z}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k,$$

then the proofs of the parts of Theorem A and Theorem C dealing with C_f go through verbatim for S_f.

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