GEOMETRIC ENTROPY AND PATTERSON-SULLIVAN CURRENTS

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Abstract. We introduce and study the notion of geometric entropy $h_T(\mu)$ for a geodesic current $\mu$ on a free group $F$ with respect to a point $T$ in the non-projectivized Outer Space $cv(F)$. Thus $T$ is an $\mathbb{R}$-tree equipped with a minimal free and discrete isometric action of $F$. The geometric entropy $h_T(\mu)$ measures the slowest exponential decay rate of the “weights” of $\mu$ on cylinder sets in $T$, with respect to the $T$-length of the segment defining such a cylinder.

We obtain an explicit formula for $h_T'(\mu_T)$, where $T, T' \in cv(F)$ are arbitrary points and where $\mu_T$ is a Patterson-Sullivan current corresponding to $T$, in terms of the volume entropy of $T$ and the extremal distortion of distances in $T$ with respect to distances in $T'$.

We conclude that for $T \in CV(F)$ (where $CV(F) \subseteq cv(F)$ is the projectivized Outer space consisting of all elements of $cv(F)$ with co-volume 1) and for a Patterson-Sullivan current $\mu_T$ corresponding to $T$ the function $CV(F) \to \mathbb{R}, \, T' \mapsto h_{T'}(\mu_T)$, achieves a strict global maximum at $T' = T$.

We also show that for any $T \in cv(F)$ and any geodesic current $\mu$ on $F$, we have $h_T(\mu) \leq h(T)$, where $h(T)$ is the volume entropy of $T$, and the equality is realized when $\mu = \mu_T$.

1. Introduction

In [17] Culler and Vogtmann introduced a free group analogue of the Teichmüller space of a hyperbolic surface, which has become known as Culler-Vogtmann’s Outer space. The Outer space proved to be a fundamental object in the study of the outer automorphism group of a free group and of individual outer automorphisms.

Let $F$ be a free group of finite rank $k \geq 2$. The nonprojectivized Outer space $cv(F)$ consists of all minimal free and discrete isometric actions of $F$ on $\mathbb{R}$-trees. Two trees in $cv(F)$ are considered equal if there exists an $F$-equivariant isometry between them. Note that for every $T \in cv(F)$ the action of $F$ on $T$ is cocompact. There are several topologies on $cv(F)$ that are all known to coincide [44]: the equivariant Gromov-Hausdorff convergence topology, the point-wise translation length function convergence topology, and the weak CW-topology (see Section 3 below for more details). There is a natural continuous left action of $Out(F)$ on $cv(F)$ that corresponds to pre-composing an action of $F$ on $T$ with the inverse of an automorphism of $F$. One often works with the projectivized version $CV(F)$ of $cv(F)$, called the Outer space, which consists of all $T \in cv(F)$ such that the quotient graph $T/F$ has volume 1. The space $CV(F)$ is a closed $Out(F)$-invariant subset of $cv(F)$.

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A geodesic current is a measure-theoretic generalization of the notion of a free homotopy class of a closed curve on a surface and of the notion of a conjugacy class in a group. The study of geodesic currents in the context of hyperbolic surfaces was initiated by Bonahon [3, 4]. Bonahon extended the notion of a geometric intersection number between two (free homotopy classes of) closed curves on a hyperbolic surface to a symmetric and mapping-class-group invariant notion of an intersection number between two geodesic currents. He also showed that the Liouville embedding of the Teichmüller space into the space of projectivized geodesic currents extends to a topological embedding of Thurston's compactification of the Teichmüller space.

Let \( \partial F \) be the hyperbolic boundary of \( F \) and let \( \partial^2 F \) be the set of all pairs \( (\xi, \zeta) \in \partial F \times \partial F \) such that \( \xi \neq \zeta \). There is a natural left translation action of \( F \) on \( \partial F \) and hence on \( \partial^2 F \). A geodesic current on \( F \) is a positive Radon measure on \( \partial^2 F \) that is \( F \)-invariant (one also sometimes requires currents to be invariant with respect to the "flip" map \( \partial^2 F \to \partial^2 F, (\xi, \zeta) \mapsto (\zeta, x_1) \), but we do not impose this restriction in this paper). The space \( \text{Curr}(F) \) of all geodesic currents on \( F \) is locally compact and comes equipped with a natural continuous action of \( \text{Out}(F) \) by linear transformations. The study of geodesic currents also proved useful in the context of free groups (see, for example, [39, 28, 29, 32, 33, 10, 11, 12, 19]). Thus in [28, 29] Kapovich constructed a canonical Bonahon-type \( \text{Out}(F) \)-invariant continuous "intersection form" \( I : \text{cv}(F) \times \text{Curr}(F) \to \mathbb{R} \). In an recent paper [33] Kapovich and Lustig extended this intersection form to the "boundary" of \( \text{cv}(F) \) and constructed its continuous \( \text{Out}(F) \)-invariant extension \( I : \overline{\text{cv}} \times \overline{\text{Curr}}(F) \to \mathbb{R} \).

Here \( \overline{\text{cv}}(F) \) is the closure of \( \text{cv}(F) \) in the equivariant Gromov-Hausdorff (or the length function) topology. It is known that \( \overline{\text{cv}}(F) \) consists precisely of all the minimal \textit{very small} isometric actions of \( F \) on \( \mathbb{R} \)-trees. The projectivization of \( \overline{\text{cv}}(F) \) gives the Thurston compactification \( \overline{\text{CV}}(F) = \text{CV}(F) \cup \partial \text{CV}(F) \) of the Outer space \( \text{CV}(F) \). Motivated by Bonahon's result, in [35] Kapovich and Nagnibeda constructed the Patterson-Sullivan map \( \tau : \text{CV}(F) \to \overline{\text{Curr}}(F) \) and proved that this map is an \( \text{Out}(F) \)-equivariant continuous embedding (here \( \overline{\text{Curr}}(F) \) is the space of \textit{projectivized} geodesic currents on \( F \)). Since \( \overline{\text{Curr}}(F) \) is compact, the closure of the image of \( \tau \) gives a compactification of \( \text{CV}(F) \). However, unlike in Bonahon's case, it turns out that this compactification is not the same as Thurston's compactification \( \overline{\text{CV}}(F) \) of \( \text{CV}(F) \). This follows from a result of Kapovich and Lustig [32] who proved that there does not exist a continuous \( \text{Out}(F) \)-equivariant map \( \partial \text{CV}(F) \to \overline{\text{Curr}}(F) \). The map \( \tau \) associates to every point \( T \in \text{CV}(F) \) the projective class of its Patterson-Sullivan current \( \mu_T \in \text{Curr}(F) \). For \( T \in \text{cv}(F) \), a Patterson-Sullivan current \( \mu_T \) corresponding to \( T \) is determined by \( T \), uniquely up to a positive scalar multiple (so that its projective class is uniquely determined by \( T \)) and is defined in terms of Patterson-Sullivan measures on \( \partial F \) corresponding to \( T \). In turn, Patterson-Sullivan measures on \( \partial F \) (which, via an \( F \)-equivariant quasiisometry between \( F \) and \( T \), can be identified with \( \partial T \)) have several equivalent definitions, including one given by a conformal density, that is a family of measures \( (\mu_x)_{x \in T} \) on \( \partial F \) satisfying explicit equations on their mutual Radon-Nykodim derivatives in terms of Busemann functions. The measures \( \mu_x \) define the same measure class on \( \partial F \). As shown by Furman [20], there is a unique (up to a scalar multiple) flip-invariant geodesic current in the measure class of \( \mu_x^2 \),
namely the current $\mu_T$. We give brief definitions of the relevant notions in Section 4 below and refer the reader to [35] for a detailed discussion.

Let $T \in cv(F)$. Recall that an $F$-equivariant quasi-isometry between $F$ and $T$ gives an identification of $\partial F$ and $\partial T$. Any current $\mu \in Curr(F)$ is uniquely determined by its values on the cylinder sets $Cyl_T([x,y)|T] \subseteq \partial^2 T$, where $[x,y)|T$ varies over all nondegenerate geodesic segments in $T$. The cylinder $Cyl_T([x,y)|T]$ consists of all $(\xi,\zeta) \in \partial^2 T$ such that the $T$-geodesic from $\xi$ to $\zeta$ passes through $[x,y)|T$. Note that since $\mu$ is $F$-invariant, the value $\mu(Cyl_T([x,y)|T])$ depends only on $\mu$ and the path which is the image of $[x,y)|T$ in the quotient graph $T/F$. It is typically the case that the "weights" $\mu(Cyl_T([x,y)|T])$ tend to 0 as $d_T(x,y) \to \infty$ and, moreover, in many interesting cases (such as that of Patterson-Sullivan currents), this convergence is exponentially fast. We introduce the notion of geometric entropy $h_T(\mu)$ of $\mu$ with respect to $T$ to measure the slowest exponential rate of decay of the weights $\mu(Cyl_T([x,y)|T])$ as $d_T(x,y)$ tends to infinity. More precisely (see Definition 5.1 below):

$$h_T(\mu) := \lim inf_{d_T(x,y) \to \infty} \frac{-\log \mu(Cyl_T([x,y)|T])}{d_T(x,y)}.$$ 

Thus if $h_T(\mu) > s > 0$ then there is $C > 0$ such that for all $x, y \in T$ with $d_T(x,y) \geq 1$ we have

$$\mu(Cyl_T([x,y)|T]) \leq C \exp(-s \cdot d_T(x,y)).$$

We first establish some basic properties of geometric entropy in Section 5. (In particular $h_T(\mu) = h_T(\mu_\mu)$ for any $c > 0$, $\mu \in Curr(F)$, so that $h_T(\mu)$ depends only on the projective class of $\mu$.) We note that for a fixed $\mu \in Curr(F)$ the function $E_\mu: cv(F) \to \mathbb{R}$, $T \mapsto h_T(\mu)$ is continuous. On the other hand, for any $T \in cv(F)$, function $h_T: Curr(F) \to \mathbb{R}$, $\mu \mapsto h_T(\mu)$ is highly discontinuous. Indeed for any "rational" current $\lambda \eta_g (\text{where } \lambda \geq 0, g \in F - \{1\})$ we have $h_T(\lambda \eta_g) = 0$. The set of all rational currents is dense in $Curr(F)$ but there are many currents with positive geometric entropy with respect to $T$. We obtain an explicit formula for the geometric entropy of a Patterson-Sullivan current $\mu_T$ of $T \in cv(F)$ with respect to an arbitrary $T' \in cv(F)$. We then solve two types of extremal problems regarding maximal values of the geometric entropy with either the tree or the current arguments fixed. Our main results are:

**Theorem A.** Let $T \in cv(F)$ and let $\mu_T \in Curr(F)$ be a Patterson-Sullivan current corresponding to $T$. Let $h(T)$ be the critical exponent of $T$. Then:

1. We have $h_T(\mu_T) = h(T)$.
2. For any $T' \in cv(F)$ we have

$$h_{T'}(\mu_T) = h(T) \inf_{g \in F - \{1\}} \frac{||g||_T}{||g||_{T'}} = \frac{h(T)}{\sup_{g \in F - \{1\}} \frac{||g||_{T'}}{||g||_T}}.$$

Here, for $f \in F$ and $T \in cv(F)$, $||f||_T := \inf_{x \in T} d_T(x, fx)$ is the translation length of $f$. Note that the critical exponent $h(T)$ coincides with the volume entropy of $T$, that is,

$$h(T) = \lim_{R \to \infty} \frac{\log \# \{g \in F : d_T(x_0, gx_0) \leq R \}}{R},$$

where $x_0 \in T$ is an arbitrary base-point.
It is also known [53, 28, 29] that
\[
\inf_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}} = \min_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}}, \quad \sup_{f \in F^{-1}} \frac{\|f\|_T}{\|f\|_{T'}} = \max_{f \in F^{-1}} \frac{\|f\|_T}{\|f\|_{T'}}
\]
and, moreover, one can algorithmically (in an appropriate sense) find \(g, f \in F\) realizing the above equalities. Using Theorem A, for any \(T\) and, moreover, one can algorithmically (in an appropriate sense) find \(E\) extremal values of the function \(E_{\mu_T}\) on \(CV(F)\) and show that this function achieves strict maximum at \(T\):

**Corollary B.** Let \(T, T' \in CV(F)\) be such that \(T \neq T'\). Let \(\mu_T \in \text{Curr}(F)\) be a Patterson-Sullivan current corresponding to \(T\) and let \(h(T)\) be the critical exponent of \(T\). Then:

1. For any \(T' \in CV(F)\) such that \(T' \neq T\) we have
   \[h_{T'}(\mu_T) < h_T(\mu_T) = h(T).\]

2. We have
   \[\inf_{T' \in CV(F)} h_{T'}(\mu_T) = 0.\]

Although, as noted above, as a function of \(\mu \in \text{Curr}(F)\), the geometric entropy function \(h_T\) is highly discontinuous, we are able to compute its maximal value:

**Theorem C.** Let \(T \in \text{cv}(F)\) and let \(h = h(T)\) be the critical exponent of \(T\).

1. Let \(\mu \in \text{Curr}(F)\), \(\mu \neq 0\) be arbitrary and let \(x \in T\) be such that \(\mu_x \neq 0\). Then
   \[h_T(\mu) \leq \text{HD}_{\partial T}(\mu_x) \leq h(T)\]
   Here \(\mu_x\) is the measure on \(\partial F\) corresponding to \(\mu\) and \(x\) (see Definition 8.1 below) and \(\text{HD}_{\partial T}(\mu_x)\) is the Hausdorff dimension of \(\mu_x\) with respect to \(\partial T\) with the metric \(d_x\) (see Section 8).

2. Let \(T' \in \text{cv}(F)\) be such that \([T'] \neq [T]\) and let \(\mu_{T'}\) be a Patterson-Sullivan current for \(T'\). Then
   \[h_T(\mu_{T'}) < h(T).\]

Here \([T]\) denotes the projective class of \(T\) that is, the set of all \(cT \in \text{cv}(F)\) where \(c > 0\). Part (1) of Theorem C implies that
\[h(T) = h_T(\mu_T) = \max_{\mu \in \text{Curr}(F) - \{0\}} h_T(\mu).\]

Combining Theorem A and Theorem C we obtain:

**Corollary D.** Let \(T, T' \in \text{cv}(F)\). Then:

1. \[\inf_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}} \leq \frac{h(T')}{h(T)} \leq \sup_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}}\]

2. Suppose that \([T] \neq [T']\). Then
   \[\inf_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}} < \frac{h(T')}{h(T)} < \sup_{g \in F^{-1}} \frac{\|g\|_T}{\|g\|_{T'}}.\]
Part (2) of Corollary D implies that if \([T] \neq [T']\) and \(h(T) = h(T')\) then
\[
\inf_{g \in F^{-1}} \frac{|g||T|}{|g||T'|} < 1 < \sup_{g \in F^{-1}} \frac{|g||T|}{|g||T'|}.
\]
Thus there exist \(g, h \in F \setminus \{1\}\) such that \(|g||T| < |g||T'|\) and \(|h||T| > |h||T'|\). This provides an analogue of a theorem of Tad White [53] who proved a similar result for \(CV(F)\), that is for the situation where points in \(cv(F)\) are normalized by co-volume (see Proposition 7.6 below). The above inequality is an analogue of White’s result for the situation where we normalize points of \(cv(F)\) by volume entropy.

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2. Geodesic currents

**Convention 2.1.** For the remainder of the paper let \(F\) be a finitely generated free group of rank \(k \geq 2\). We will denote by \(\partial F\) the space of ends of \(F\) with the standard ends-space topology. Thus \(\partial F\) is a topological space homeomorphic to the Cantor set. We shall also think about \(\partial F\) as the hyperbolic boundary of \(F\), endowed with the canonical boundary topology, in the sense of the theory of word-hyperbolic groups (see, for example [21]).

We set
\(
\partial^2 F := \{ (\zeta, \xi) : \zeta, \xi \in \partial F \text{ and } \zeta \neq \xi \}.
\)

**Definition 2.2 (Geodesic currents).** A geodesic current on \(F\) is a positive, finite invariant Borel measure on \(\partial^2 F\). We denote the space of all geodesic currents on \(F\) by \(\text{Curr}(F)\). The space \(\text{Curr}(F)\) comes equipped with the weak*- topology of pointwise convergence of integrals of continuous functions (see a more explicit description of this topology below) which makes \(\text{Curr}(F)\) into a locally compact topological space.

**Definition 2.3 (Projectivized currents).** We say that two nonzero geodesic currents are equivalent, denoted \(\nu_1 \sim \nu_2\), if there exists a positive scalar \(r \in \mathbb{R}\) such that \(\nu_2 = r \nu_1\). We consider also the space
\[
\mathbb{P}\text{Curr}(F) := \{ \nu \in \text{Curr}(F) : \nu \neq 0 \} / \sim
\]
of projectivized geodesic currents on \(F\), endowed with the quotient topology. We denote the \(\sim\)-equivalence class of a nonzero geodesic current \(\nu\) by \([\nu]\).

The space \(\text{Curr}(F)\) comes equipped with a natural \(\mathbb{R}_{>0}\)-linear structure, where the operations of addition and multiplication by a scalar are continuous. Also, the group \(\text{Aut}(F)\) has a natural continuous linear left action on \(\text{Curr}(F)\) defined as follows. Let \(\psi \in \text{Aut}(F)\). Then \(\psi\) is a quasi-isometry of \(F\) (with any fixed word metric) and thus extends to a homeomorphism \(\partial \psi : \partial F \to \partial F\). Hence \(\psi\) defines a homeomorphism \(\partial^2 \psi : \partial^2 F \to \partial^2 F\), \((\zeta, \xi) \mapsto (\partial \psi(\zeta), \partial \psi(\xi))\). For a current \(\nu \in \text{Curr}(F)\) and a Borel set \(S \subseteq \partial^2 F\) put \((\psi \nu)(S) := \nu \left( (\partial^2 \psi)^{-1} S \right)\). It is not hard to check [29] that \(\psi \nu\) is \(F\)-invariant, so that \(\psi \nu \in \text{Curr}(F)\). Moreover, the subgroup \(\text{Inn}(F) \leq \text{Aut}(F)\) is contained in the kernel of this action, which therefore factors through to a continuous action of \(\text{Out}(F)\) on \(\text{Curr}(F)\) by linear transformations. This action in turn factors to an action of \(\text{Out}(F)\) on \(\mathbb{P}\text{Curr}(F)\) as well. We refer the reader to [29] for a more detailed discussion.
\textbf{Definition 2.4 (Cylinder sets).} Let $T$ be an $\mathbb{R}$-tree equipped with a nontrivial minimal free and discrete isometric action of $F$ (hence this action is co-compact). Note that $T$ is a proper Gromov-hyperbolic geodesic metric space. Denote by $\partial T$ the hyperbolic boundary of $T$ and by $\partial^2 T$ the set of all pairs $(\xi, \zeta) \in \partial^2 T$ such that $\xi \neq \zeta$. Thus for any $(\xi, \zeta) \in \partial^2 T$ there exists a unique bi-infinite (non-parameterized) oriented geodesic line $[\xi, \zeta]_T \subseteq T$ in $T$ from $\xi$ to $\zeta$. We think of $[\xi, \zeta]_T \subseteq T$ as the image of an isometric embedding from $\mathbb{R}$ to $T$, with the correct choice of an orientation on $[\xi, \zeta]_T$.

Let $x, y \in T, x \neq y$. Denote

\[Cyl_T([x, y]_T) := \{(\xi_1, \xi_2) \in \partial^2 T : [x, y]_T \subseteq [\xi_1, \xi_2]\}
\]

and the orientations on $[x, y]$ and on $[\xi_1, \xi_2]$ agree.

We call $Cyl_T([x, y]_T)$ the two-sided cylinder set corresponding to $[x, y]_T$.

It is easy to see that for every $x, y \in T, x \neq y$ the cylinder $Cyl_T([x, y]_T) \subseteq \partial^2 T$ is a compact-open subset of $\partial^2 T$. Moreover, the collection of all such cylinder sets forms a basis of the topology for $\partial^2 T$.

\textbf{Convention 2.5.} Let $T$ be as in Definition 2.4. Since $F$ acts discretely, isometrically and co-compactly on $T$, the orbit map (for any basepoint in $T$) defines a quasi-isometry $q_T : F \to T$ (where $F$ is taken with any word metric) and hence a canonical $F$-equivariant homeomorphism $\partial q_T : F \to \partial T$. It turns, $\partial q_T$ defines an $F$-equivariant homeomorphism $\partial^2 q_T : \partial^2 F \to \partial^2 T$.

We will use the homeomorphisms $\partial q_T$ and $\partial^2 q_T$ to identify $\partial F$ with $\partial T$ and, similarly, $\partial^2 F$ with $\partial^2 T$. We will often suppress this explicit identification.

Thus we will write $Cyl_T([x, y]_T) \subseteq \partial^2 F$ to mean $(\partial^2 q_T)^{-1} (Cyl_T([x, y]_T)) \subseteq \partial^2 F$. Similarly, for $\mu \in \text{Curr}(F)$, we will write $\mu(Cyl_T([x, y]_T))$ to mean $\mu((\partial^2 q_T)^{-1} (Cyl_T([x, y]_T)))$.

Moreover, if $\nu$ is an $F$-invariant positive Radon measure on $\partial^2 T$, then, using the identification $\partial^2 q_T$ of $\partial^2 F$ and $\partial^2 T$, $\nu$ defines an $F$-invariant positive Radon measure $\nu'$ on $\partial^2 T$ on $\partial^2 F$, that is a geodesic current $\nu' \in \text{Curr}(F)$. We will often regard $\nu$ and $\nu'$ as being the same and write $\nu(Cyl_T([x, y]_T))$ whereas we mean $\nu' \in \text{Curr}(F)$.

\textbf{Remark 2.6.} Let $T, T' \subset \text{cv}(F)$ be arbitrary. Let $\phi : T \to T'$ be an $F$-equivariant quasi-isometry and let $\partial^2 \phi : \partial^2 T \to \partial^2 T'$ be the homeomorphism induced by $\phi$. Then $\partial^2 \phi$ is canonical, that is, it does not depend on the choice of an $F$-equivariant quasi-isometry between $T$ and $T'$ but is uniquely determined by $T$ and $T'$ themselves. Moreover, if $S \subseteq \partial^2 T$ then, under the identifications of $\partial^2 T$ and $\partial^2 T'$ with $\partial^2 F$ in Convention 2.5, the sets $S$ and $\partial^2 \phi(S)$ correspond to the same subset of $\partial^2 F$.

\textbf{Convention 2.7.} For a (finite or infinite) graph $\Delta$, denote by $V \Delta$ the set of all vertices of $\Delta$, and denote by $E \Delta$ the set of all oriented edges of $\Delta$. Combinatorially, we use Serre’s convention regarding graphs. Thus $\Delta$ comes equipped with a function $-1 : E \Delta \to E \Delta$, such that $e^{-1} \neq e$, $(e^{-1})^{-1} = e$ for every $e \in E \Delta$ and such that $o(e) = t(e^{-1})$ and $t(e) = o(e^{-1})$ for every $e \in E \Delta$. The edge $e^{-1}$ is called the inverse of $e$. 
An edge-path $\gamma$ in $\Delta$ is a sequence of oriented edges which connects a vertex $o(\gamma)$ (origin) with a vertex $t(\gamma)$ (terminus). A path is called reduced if it does not contain a back-tracking, that is a path of the form $ee^{-1}$, where $e \in E\Delta$.

If $\gamma = e_1, \ldots, e_n$ is an edge-path in $\Delta$, where $e_i \in E\Delta$, we call $n$ the simplicial length of $\gamma$ and denote it by $|\gamma|$.

An orientation on $\Delta$ is a partition $E\Delta = E^+\Delta \sqcup E^-\Delta$, where for every $e \in E\Delta$ one of the edges $e, e^{-1}$ belongs to $E^+\Delta$ and the other edge belongs to $E^-\Delta$.

We denote by $\mathcal{P}(\Delta)$ the set of all finite reduced edge-paths in $\Delta$. For a vertex $x \in V\Delta$, we denote by $\mathcal{P}_x(\Delta)$ the collection of all $\gamma \in \mathcal{P}(\Delta)$ that begin with $x$.

**Definition 2.8** (Simplicial charts). Let $\Gamma$ be a finite connected graph without degree-one and degree-two vertices such that $\pi_1(\Gamma) \cong F$. Let $\alpha : F \to \pi_1(\Gamma, p)$ be an isomorphism, where $p$ is a vertex of $\Gamma$. We call such $\alpha$ a simplicial chart or a marking for $F$.

**Convention 2.9.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart. We consider $X := \tilde{\Gamma}$, a topological tree, and denote the covering map from $X$ to $\gamma$ by $\tilde{j} : X \to \Gamma$.

1. For $\gamma \in \mathcal{P}(X)$ we call the reduced edge-path $\tilde{j}(\gamma)$ in $\Gamma$ the label of $\gamma$. As there is only one reduced edge-path connecting two arbitrary vertices in a tree, we will often write $[x, y] = [x, y]\big|_X$ for a path in $X$ with origin $x$ and terminus $y$.

2. Let $\partial X$ denote the space of ends of $X$ with the natural ends-space topology. Suppose we endow $\Gamma$ with a metric graph structure (see Definition 3.4 below) by giving each topological edge of $\Gamma$ a positive length. This metric graph structure lifts to $X$ and turns $X$ into an $\mathbb{R}$-tree $T$, endowed with a minimal free and discrete isometric action of $F$ via $\alpha$. Note that $\partial^2 T$ is canonically homeomorphic to the space of ends of $X = \tilde{\Gamma}$ and this homeomorphism does not depend on the choice of a metric structure on $\Gamma$.

We get $F$-equivariant homeomorphisms $\partial T : \partial F \to \partial T$ and $\partial^2 q_T : \partial^2 F \to \partial^2 T$, as described in Convention 2.5 above. A crucial feature of this construction is that $\partial q_T$ and $\partial^2 q_T$ do not depend on the choice of a metric structure on $\Gamma$.

Note also, that for any points $x, y \in X$ the geodesic segment $[x, y]\big|_T$, considered as a subset of $X$, does not depend on the choice of a metric structure on $\Gamma$. Similarly, the cylinder $Cyl_X([x, y]) \subseteq \partial^2 T = \partial^2 X$ does not depend on the choice of a metric structure on $\Gamma$ either.

Thus for $x, y \in X$, $x \neq y$, we will write $Cyl_X([x, y]) \subseteq \partial^2 X$ for $Cyl_T([x, y]\big|_T)$, where $T$ comes from any metric structure on $\Gamma$ (say, where all edges of $\Gamma$ have length 1).

Recall, that, as explained in Convention 2.5, we will usually think of $Cyl_X([x, y])$ as a subset of $\partial^2 F$, using the identification $\partial^2 q_T$.

Let $\gamma \in \mathcal{P}(X)$ and $x = o(\gamma)$, $y = t(\gamma)$, so that $\gamma = [x, y]$. We denote $Cyl_X(\gamma) := Cyl_X([x, y])$.

Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart, let $X := \tilde{\Gamma}$ and let $j : X \to \Gamma$ be the covering. It is easy to see that for $\nu_n, \nu \in Curr(F)$ we have $\lim_{n \to \infty} \nu_n = \nu$ if and only if $\lim_{n \to \infty} \nu_n(Cyl_X(\gamma)) = \nu(Cyl_X(\gamma))$ for every $\gamma \in \mathcal{P}(X)$. Moreover, for $\nu, \nu' \in Curr(F)$ we have $\nu = \nu'$ iff $\nu(Cyl_X(\gamma)) = \nu'(Cyl_X(\gamma))$ for every $\gamma \in \mathcal{P}(X)$.

**Notation 2.10.** Note that for any $f \in F$ and $\gamma \in \mathcal{P}(X)$ we have $fCyl_X(\gamma) = Cyl_X(f\gamma)$. Since geodesic currents are, by definition, $F$-invariant, for a geodesic current $\gamma$.
current ν and for γ ∈ ℙ(X) the value ν(∅γX(γ)) only depends on the label j(γ) ∈ ℙ(Γ) of γ.

For this reason for any reduced edge-path v ∈ ℙ(Γ) we denote by ⟨v, ν⟩ the value ν(∅γX(γ)) where γ ∈ ℙ(X) is any reduced edge-path with label v.

3. The Culler-Vogtmann Outer Space

The Culler-Vogtmann outer space, introduced by Culler and Vogtmann in a seminal paper [17], is a free group analogue of the Teichmüller space of a closed surface of negative Euler characteristic. We refer the reader to the original paper [17] and to a survey paper [52] for a detailed discussion of the basic facts listed in this section and for the further references.

Definition 3.1 (Non-projectivized Outer Space). Let F be a finitely generated free group of rank k ≥ 2.

The non-projectivized outer space cv(F) consists of all minimal free and discrete isometric actions of F on ℝ-trees. Two such trees T1, T2 are considered equal in cv(F) if there exists an F-equivariant isometry between them. The space cv(F) is endowed with the equivariant Gromov-Hausdorff convergence topology.

For T ∈ cv(F) and c > 0 denote by cT ∈ cv(F) the tree that coincides with T as a topological space and has the same F-action, but where the metric is multiplies by c.

It turns out that every T ∈ cv(F) is uniquely determined by its translation length function ℓT : F → ℝ, where for every g ∈ F

$$\ell_T(g) = ||g||_T = \min_{x \in T} d_T(x, gx)$$

is the translation length of g. Note that ℓT(g) = ℓT(hgh⁻¹) for every g, h ∈ F. Thus ℓT can be thought of as a function on the set of conjugacy classes in F. The space cv(F) comes equipped with a natural left Out(F)-action by homeomorphisms. At the length function level, if φ ∈ Out(F), T ∈ cv(F) and g ∈ F we have

$$\ell_{φT}([g]) = ℓ_T(φ^{-1}[g]).$$

It is known that the equivariant Gromov-Hausdorff topology on cv(F) coincides with the pointwise convergence topology at the level of length functions. Thus for Tn, T ∈ cv(F) we have limn→∞ Tn = T if and only if for every g ∈ F we have limn→∞ ℓTn(g) = ℓT(g).

Definition 3.2 (Projectivized Outer Space). Denote by CV(F) the subset of cv(F) consisting of all T ∈ cv(F) such that the quotient graph T/F has volume 1.

The space CV(F) is a closed Out(F)-invariant subset of cv(F) and it is called the projectivized Outer Space of F.

It is known that CV(F) is canonically homeomorphic to cv(F)/∼, where T1 ∼ T2 if there is c > 0 such that T2 = cT1. This fact justifies the term “projectivized Outer Space”. For T ∈ cv(F) we denote by [T] the ∼-equivalence class of T and call [T] the projective class of T.

Points of cv(F) have a more explicit combinatorial description as “marked metric graph structures” on F:
Definition 3.3 (Metric graph structure). Let $\Gamma$ be a finite connected graph without degree-one and degree-two vertices. A metric graph structure $\mathcal{L}$ on $\Gamma$ is a function $\mathcal{L} : E\Gamma \to \mathbb{R}$ such that for every $e \in E\Gamma$ we have

$$\mathcal{L}(e) = \mathcal{L}(e^{-1}) > 0.$$ 

A semi-metric graph structure $\mathcal{L}$ on $\Gamma$ is a function $\mathcal{L} : E\Gamma \to \mathbb{R}$ such that for every $e \in E\Gamma$ we have

$$\mathcal{L}(e) = \mathcal{L}(e^{-1}) \geq 0.$$ 

A semi-metric graph structure $\mathcal{L}$ on $\Gamma$ is nondegenerate if there exists a subforest $Z$ in $\Gamma$ such that $\mathcal{L}(e) > 0$ for every $e \in E\Gamma - EZ$.

For a semi-metric graph structure $\mathcal{L}$ on $\Gamma$ define the volume of $\mathcal{L}$ as

$$\text{vol}(\mathcal{L}) = \sum_{e \in E^{+}\Gamma} \mathcal{L}(e)$$

where $E\Gamma = E^{+}\Gamma \cup E^{-}\Gamma$ is any orientation on $\Gamma$.

If $\mathcal{L}$ is a semi-metric graph structure on $\Gamma$ and $v = e_1, \ldots, e_n$ is an edge-path in $\Gamma$, we denote

$$\mathcal{L}(v) = \sum_{i=1}^{n} \mathcal{L}(e_i)$$

and call $\mathcal{L}(v)$ the $\mathcal{L}$-length of $v$.

Definition 3.4 (Marked metric graph structure). Let $F$ be a free group of finite rank $k \geq 2$. A marked metric graph structure on $F$ is a pair $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is a simplicial chart for $F$ and $\mathcal{L}$ is a metric structure on $\Gamma$.

A marked semi-metric graph structure on $F$ is a pair $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is a simplicial chart for $F$ and $\mathcal{L}$ is a semi-metric structure on $\Gamma$. A marked semi-metric graph structure $(\alpha, \mathcal{L})$ is non-degenerate if $\mathcal{L}$ is nondegenerate.

Convention 3.5. Let $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$, be a marked metric graph structure on $F$. Then $(\alpha, \mathcal{L})$ defines a point $T$ as follows. Topologically, let $T = \tilde{\Gamma}$, with an action of $F$ on $T$ via $\alpha$. We lift the metric structure $\mathcal{L}$ from $\Gamma$ to $T$ by giving every edge in $T$ the same length as that of its projection in $\Gamma$. This makes $T$ into an $\mathbb{R}$-tree equipped with a minimal free and discrete isometric action of $F$. Thus $T \in cv(F)$ and in this situation we will sometimes use the notation $T = (\alpha, \mathcal{L}) \in cv(F)$. Note that $T/F = \Gamma$. Moreover, it is not hard to see that every point of $cv(F)$ arises in this fashion and that $CV(F)$ is exactly the set of all those $T = (\alpha, \mathcal{L}) \in cv(F)$ where $(\alpha, \mathcal{L})$ is a marked metric graph structure on $F$ with $\text{vol}(\mathcal{L}) = 1$.

Remark 3.6. Let $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$, be a nondegenerate marked metric graph structure on $F$. Then $(\alpha, \mathcal{L})$ also defines a point $T$ in $cv(F)$ as follows. First, collapse to points all edges $e \in \Gamma$ with $\mathcal{L}(e) = 0$. This produces a graph $\Gamma'$ with a metric structure $\mathcal{L}'$. Moreover, since $\mathcal{L}$ was nondegenerate, the marking $\alpha : F \to \pi_1(\Gamma, p)$ induces a marking $\alpha' : F \to \pi_1(\Gamma', p')$ where $p' \in \tilde{\Gamma}'$ is the image of $p$ in $\Gamma'$. Let $T = (\alpha', \mathcal{L}') \in cv(F)$ be the $\mathbb{R}$-tree constructed as in Convention 3.5. We say that $T$ is the point of $cv(F)$ defined by $(\alpha, \Gamma)$ and sometimes use the notation $T = (\alpha, \Gamma) \in cv(F)$. 


Lemma 3.9. Let \( \lambda : F \to \pi_1(\Gamma, p) \) be a simplicial chart for \( F \). Fix an orientation \( ET = E^+\Gamma \cup E^-\Gamma \) on \( \Gamma \) and let \( E^+\Gamma = \{e_1, \ldots, e_m\} \), where \( m = \#E^+\Gamma \).

Let \( V_\alpha \subseteq cv(F) \) be the set of all \( T = (\alpha, \mathcal{L}) \) where \( \mathcal{L} \) is a non-degenerate semi-metric structure on \( \Gamma \). Let \( U_\alpha \) be the set of all \( T = (\alpha, \mathcal{L}) \) where \( \mathcal{L} \) is a metric structure on \( \Gamma \). Thus \( U_\alpha \subseteq V_\alpha \). We call \( V_\alpha \) the elementary chart corresponding to \( \alpha \) and we call \( U_\alpha \) the elementary open chart corresponding to \( \alpha \).

There is a natural map \( \lambda_\alpha : V_\alpha \to \mathbb{R}^m \) defined as \( \lambda_\alpha(\alpha, \mathcal{L}) = (\mathcal{L}(e_1), \ldots, \mathcal{L}(e_m)) \). It is known that \( \lambda_\alpha : V_\alpha \to \mathbb{R}^m \) is injective and is a homeomorphism onto its image. In particular, \( \lambda_\alpha(U_\alpha) \) is the positive open cone in \( \mathbb{R}^m \), that is, \( \lambda_\alpha(U_\alpha) \) consists of all points in \( \mathbb{R}^m \) all of whose coordinates are positive. Therefore \( U_\alpha \) is homeomorphic to an open cone in \( \mathbb{R}^m \).

The space \( cv(F) \) is the union of open cones \( U_\alpha \) taken over all simplicial charts \( \alpha \) on \( F \). Moreover, every point \( T \in cv(F) \) belongs to only finitely many of the elementary charts \( V_\alpha \). It is also known that the standard topology on \( cv(F) \) coincides with the weakest topology for which all the maps \( \lambda_\alpha^{-1} : \lambda_\alpha(V_\alpha) \to V_\alpha \) are continuous.

Notation 3.8. Denote \( F^\times = F - \{1\} \).

We will need the following basic lemma which shows that for \( T, T' \in cv(F) \) extremal distortions of the translation length functions for \( T \) and \( T' \) give the optimal stretching constants for \( F \)-equivariant quasi-isometries between \( T \) and \( T' \).

Lemma 3.9. Let \( T, T' \in cv(F) \) and let \( \phi : T \to T' \) be an \( F \)-equivariant quasi-isometry. Then the following hold:

1. \( \sup_{g \in F^\times} \frac{\|g\|_{T'}}{\|g\|_T} = \limsup_{d_T(x,y) \to \infty} \frac{d_T'(\phi(x), \phi(y))}{d_T(x,y)} \).
2. \( \inf_{g \in F^\times} \frac{\|g\|_{T'}}{\|g\|_T} = \liminf_{d_T(x,y) \to \infty} \frac{d_T'(\phi(x), \phi(y))}{d_T(x,y)} \).
3. There is \( C > 0 \) such that for any \( x, y \in T \) we have
   \[ \lambda_1 d_T(x,y) - C \leq d_T'(\phi(x), \phi(y)) \leq \lambda_2 d_T(x,y) + C, \]
   where \( \lambda_1 = \inf_{g \in F^\times} \frac{\|g\|_{T'}}{\|g\|_T}, \) \( \lambda_2 = \sup_{g \in F^\times} \frac{\|g\|_{T'}}{\|g\|_T} \).

Proof. Let \( x_0 \in V_T \) be a base-point and let \( x'_0 = \phi(x_0) \in V_{T'} \). Thus \( \phi(gx_0) = gx'_0 \) for all \( g \in F \).

1. For any \( g \in F^\times \) we have \( \lim_{n \to \infty} \frac{d_T(x_0, g^nx_0)}{n} = \|g\|_T \), \( \lim_{n \to \infty} \frac{d_T'(x'_0, g^n x'_0)}{n} = \|g\|_{T'} \)
   and hence \( \lim_{n \to \infty} \frac{d_T(x_0, g^n x_0)}{d_T'(x'_0, g^n x'_0)} = \|g\|_T / \|g\|_{T'} \).

Then \( \frac{\|g\|_{T'}}{\|g\|_T} \leq \limsup_{d_T(x,y) \to \infty} \frac{d_T'(\phi(x), \phi(y))}{d_T(x,y)} \) and so
Suppose now that $d_T(p_n, q_n) \to \infty$ and

$$\lim_{n \to \infty} \frac{d_T(p_n, q_n)}{d_T(p_n, q_n)} = \limsup_{d_T(x, y) \to \infty} \frac{d_T(x, y)}{d_T(x, y)}.$$

There is a constant $M = M(T') \geq 1$ such that there are some $g_n, h_n \in F$ with $d_T(p_n, g_n x_0), d_T(q_n, h_n x_0) \leq M$ and such that the geodesic $[g_n x_0, h_n x_0]_T$ projects to a closed cyclically reduced path in $T/F$. That is, the points $g_n x_0, h_n x_0$ belong to the axis of the element $h_n g_n^{-1}$ and $d_T(g_n x_0, h_n x_0) = ||h_n g_n^{-1}||_T$. Translating $[p_n, q_n]$ by $g_n^{-1}$ we may assume that $g_n = 1$ for every $n \geq 1$. Thus $d(x_0, h_n x_0) = ||h_n||_T$ and $d_T(p_n, q_n), d_T(g_n, h_n x_0) \leq M$. Hence $|d_T(p_n, q_n) - ||h_n||_T| \leq 2M$. Note that $x_0$ and $h_n x_0$ belong to the axis of $h_n$ in $T$.

Denote $p'_n = \phi(p_n), q'_n = \phi(q_n)$. Since $\phi$ is an $F$-equivariant quasi-isometry, the $\phi$-image of the axis of $h_n$ in $T$ is an $h_n$-invariant quasigeodesic in $T'$ which is at a bounded Hausdorff distance from an $h_n$-invariant geodesic in $T'$, that is, from the axis of $h_n$ in $T'$. Hence there is some constant $C \geq 1$ such that $|d_T(p'_n, q'_n) - ||h_n||_T| \leq C$.

Therefore

$$\lim_{n \to \infty} \frac{d_T(p_n, q_n)}{d_T(p_n, q_n)} \leq \lim_{n \to \infty} \frac{||h_n||_T + M}{||h_n||_T - C} \leq \sup_{g \in F} \frac{||g||_T}{||g||_T}.$$

Hence

$$\sup_{g \in F} \frac{||g||_T}{||g||_T} = \limsup_{d_T(x, y) \to \infty} \frac{d_T(x, y)}{d_T(x, y)},$$

as required.

Part (2) is established using a similar argument to part (1) and we omit the details.

For part (3), denote $\lambda_1 = \inf_{g \in F} \frac{||g||_T}{||g||_T}$ and $\lambda_2 = \sup_{g \in F} \frac{||g||_T}{||g||_T}$.

Let $x, y \in T$ be arbitrary and let $x' = \phi(x), y' = \phi(y)$. As in the proof of (1), there exist elements $g, h \in F$ such that such that $d_T(x, g x_0), d_T(y, h x_0) \leq M$ and such that the geodesic $[g x_0, h x_0]_T$ projects to a closed cyclically reduced path in $T/F$. That is, the points $g x_0, h x_0$ belong to the axis of the element $h g^{-1}$ and $d_T(g x_0, h x_0) = ||h g^{-1}||_T$. In view of $F$-equivariance of $\phi$ we may assume that $g = 1$. Thus $d_T(x, x_0), d_T(y, h x_0) \leq M$, the points $x_0, h x_0$ belong to the axis of the element $h$ and $d_T(x_0, h x_0) = ||h||_T$. Since $\phi$ is an $F$-equivariant quasi-isometry, it follows that

$$|d_T(x', y') - ||h||_T| \leq C,$$

for some constant $C > 0$ independent of $x, y$. Note that by definition of $\lambda_2$ we have $||h||_T \leq \lambda_2 ||h||_T$. Then

$$d_T(x', y') \leq ||h||_T + C \leq \lambda_2 ||h||_T + C \leq \lambda_2 d_T(x, y) + 2M + C = \lambda_2 d_T(x, y) + (2M + C),$$

as required. The proof that $d_T(x', y') \geq \lambda_1 d_T(x, y) - (2M + C)$, is similar, and we omit the details.

□
4. Patterson-Sullivan currents and the Outer space

**Definition 4.1** (Volume Entropy). Let $T \in cv(F)$. The volume entropy $h = h(T)$ is defined as

$$h(T) = \lim_{R \to \infty} \log \# \{ g \in F : d_T(x_0, gx_0) \leq R \},$$

where $x_0 \in T$ is a basepoint. It turns out that the above limit always exists and does not depend on the choice of a basepoint $x_0 \in X$.

The following facts are well-known:

**Proposition 4.2.** Let $T \in cv(F)$. Then:

1. We have $h(T) > 0$.
2. For any $x_0 \in X$ we have

$$h(T) = \lim_{R \to \infty} \frac{\log \# \{ x \in VT : d_T(x_0, x) \leq R \}}{R}.$$

3. The Poincaré series of $X$ with respect to a base-point $x_0 \in X$ is

$$\Pi_{x_0}(s) := \sum_{g \in F} e^{-s} d(x_0, gx_0).$$

Then the volume entropy $h = h(T)$ is exactly the emph{critical} exponent of the Poincaré series. Namely, $\Pi_{x_0}(s)$ converges for all $s > h$ and diverges for all $s < h$.

The following is a special case of a result of [20]:

**Proposition-Definition 4.3.** Let $T \in cv(F)$ and let $h = h(T)$ be the critical exponent. Then for every $x_0 \in X$ any weak limit $\mu$, as $s \to h^+$, of the probability measures

$$\frac{1}{\Pi_{x_0}(s)} \sum_{g \in F} e^{-s} d(x_0, gx_0) \text{Dirac}(gx_0).$$

is a measure supported at $\partial T$. Moreover, the measure-class of $\mu$ is uniquely defined and does not depend on the choice of $x_0$ or on the choice of a weak limit.

We call any such $\mu$ a Patterson-Sullivan measure on $\partial T = \partial F$ corresponding to $T \in cv(F)$.

There is an alternative description of Patterson-Sullivan measures in terms of conformal densities but we will not need it in this paper. We refer the reader to [35] for a detailed discussion on the topic.

The following statement is essentially a corollary of Proposition 1 of Furman [20].

**Proposition-Definition 4.4** (Patterson-Sullivan current). Let $T \in cv(F)$. Let $\mu$ be a Patterson-Sullivan measure on $\partial T = \partial F$ (where we are using the identification of $\partial F$ and $\partial T$ as in Convention 2.5).

Then there exists a unique, up to a scalar multiple, $F$-invariant and flip-invariant nonzero locally finite measure $\nu$ on $\partial^2 X$ in the measure class of $\mu \times \mu$.

Such a measure $\nu$ is called a Patterson-Sullivan current for $T \in cv(F)$. Since $\nu$ is unique up to a scalar multiple, its projective class $[\nu]$ is called the projective Patterson-Sullivan current corresponding to $T \in cv(F)$.

Moreover, Furman’s results [20] imply that for $T \in cv(F)$ the projective Patterson-Sullivan current corresponding to $T$ depends only on the projective class $[T]$ of $T$.  

5. Geometric entropy of a current

Recall that in view of Convention 2.5, for any \( T \in \text{cv}(F) \) and any \( x, y \in T \), \( x \neq y \), we have a well-defined cylinder set \( Cyl_T([x, y]|T) \subseteq \partial^2 F \).

**Definition 5.1** (Geometric entropy of a current). Let \( \mu \in \text{Curr}(F) \) and let \( T \in \text{cv}(F) \). Define the geometric entropy of \( \mu \) with respect to \( T \) as

\[
  h_T(\mu) := \liminf_{d_T(x, y) \to \infty} \frac{-\log \mu(Cyl_T([x, y]|T))}{d_T(x, y)}.
\]

If \( \mu(Cyl_T([x, y]|T)) = 0 \), we interpret \( \log 0 = -\infty \). Thus for \( \mu = 0 \) any any \( T \in \text{cv}(F) \) we have \( h_T(\mu) = \infty \). For any \( \mu \in \text{Curr}(F) \), \( \mu \neq 0 \) and any \( T \in \text{cv}(F) \) we have \( 0 \leq h_T(\mu) < \infty \). Informally, \( h_T(\mu) \) measures the slowest exponential decay rate (with respect to \( d_T(x, y) \)) of the “weights” \( \mu(Cyl_T([x, y]|T)) \) as \( d_T(x, y) \to \infty \).

The following is a more combinatorial interpretation of the geometric entropy that follows immediately from unraveling the definitions:

**Proposition 5.2.** Let \( \mu \in \text{Curr}(F) \), \( \mu \neq 0 \) and let \( T \in \text{cv}(F) \) be determined by the pair \((\alpha, L)\), where \( \alpha : F \to \pi_1(\Gamma, p) \) is an isomorphism and \( L \) is a metric graph structure on \( \Gamma \).

Then:

1. We have

\[
  h_T(\mu) := \liminf_{|v| \to \infty} \frac{-\log(v, \mu)}{L(v)} = \liminf_{|v| \to \infty} \frac{-\log(v, \mu)_\alpha}{L(v)}.
\]

2. If \( h_T(\mu) > s \geq 0 \) then there exists \( C > 0 \) such that for every \( v \in \mathcal{P}(\Gamma) \) we have

\[
  \langle v, \mu \rangle_\alpha \leq C \exp(-sL(v)).
\]

We already noted that if \( \mu = 0 \), we get \( h_T(\mu) = \infty \) for any \( T \in \text{cv}(F) \). On the other hand, for any \( g \in F \), \( g \neq 1 \) and any \( T \in \text{cv}(F) \) we have \( h_T(\eta_g) = 0 \).

The following statement is an immediate corollary of the definition of geometric entropy:

**Lemma 5.3.** Let \( \mu \in \text{Curr}(F) \), \( \mu \neq 0 \) and let \( T \in \text{cv}(F) \). Then:

1. For any \( c > 0 \) we have \( h_T(\mu) = h_T(\mu C) \).
2. For any \( c > 0 \) we have \( h_T(\mu) = \frac{1}{c} h_T(\mu) \).

Thus \( h_T(\mu) \) depends only on the projective class \([\mu] \in \mathcal{P}\text{Curr}(F)\) of \( \mu \).

**Proposition 5.4.** Let \( T, T' \in \text{cv}(F) \). Let \( \phi : T \to T' \) be an \( F \)-equivariant quasi-isometry. There exists an integer \( M = M(\phi) \geq 1 \) with the following property.

Let \( x_1, x_2 \in T \) and let \( x_1' = \phi(x_1), x_2' = \phi(x_2) \). Let \( y_1, y_2 \in [x_1', x_2']_{T'} \) be such that \( d_{T'}(x_1', y_1) = d_{T'}(x_2', y_2) = M \). Then

\[
  \phi(Cyl_T([x_1, x_2])) \subseteq Cyl_{T'}([y_1, y_2]).
\]
Proof. This statement is a straightforward consequence of the "Bounded Cancellation Lemma" [13] for quasi-isometries between Gromov-hyperbolic spaces. Indeed, let \([x, y] \subseteq T\) be a geodesic segment in \(T\) and let \(\gamma\) be a geodesic ray in \(T\) with initial point \(y\), such that the path \([x, y] + \gamma\) is a geodesic (that is, there is no cancellation between \([x, y]\) and \(\gamma\). There is a unique geodesic ray \(\gamma'\) in \(T'\) starting at \(\phi(y)\) such that \(\gamma'\) is at a finite Hausdorff distance from \(\phi(\gamma)\). The "Bounded Cancellation Lemma" implies that the cancellation between \([\phi(x), \phi(y)]\) and \(\gamma'\) is bounded by some constant \(M \geq 1\) depending only of the quasi-isometry \(\phi\). It is not hard to check that this constant \(M\) satisfies the requirements the proposition and we leave the details to the reader. \(\square\)

**Proposition 5.5.** Let \(T, T' \in cv(F)\). Let \(\phi : T \rightarrow T'\) be an \(F\)-equivariant quasi-isometry. There exists an integer \(M_1 = M_1(\phi) \geq 1\) with the following property.

Let \(x_1, x_2 \in T\) and let \(x'_1 = \phi(x_1), x'_2 = \phi(x_2)\).

Then there exist points \(p_1, p_2, q_1, q_2 \in T'\) such that \([p_1, p_2] = [x'_1, x'_2] \cap [q_1, q_2]\), such that \(d(q_i, x'_i) \leq M_1\) and such that

\[
Cyl_{T'}([q_1, q_2]) \subseteq \phi(Cyl_T([x_1, x_2])
\]

Proof. Let \(\psi : T' \rightarrow T\) be an \(F\)-equivariant quasi-isometry which is a quasi-inverse of \(\phi\). Let \(M = M(\psi) > 0\) be the constant provided by Proposition 5.4 for \(\psi\).

Choose \(\xi, \zeta \in \partial T\) such that the bi-infinite geodesic \([x_1, x_2]_T \subseteq [\xi, \zeta]_T\). Then \(x'_1 = \phi(x'_1)\) and \(x_2 = \phi(x'_2)\) are at distance at most \(C_1 = C_1(\phi)\) from \([\phi(\xi), \phi(\zeta)]_T\). Thus \([\phi(\xi), \phi(\zeta)]_T \cap [x'_1, x'_2] = [p_1, p_2]\), where \(d(x'_1, p_1) \leq C_1\).

Recall that \(\psi\) is a quasi-isometry that is a quasi-inverse of \(\phi\), so that \(\psi(\phi(\xi)) = \xi\) and \(\psi(\phi(\zeta)) = \zeta\). Note also that \([x_1, x_2] \subseteq [\xi, \zeta] = [\psi(\phi(\xi)), \psi(\phi(\zeta))]\) and that \([p_1, p_2] \subseteq [\phi(\xi), \phi(\zeta)]\). Therefore there is some \(C_2 = C_2(\phi) > 0\) such that if \(q_1 \in [\phi(\xi), p_1]_T\), \(q_2 \in [p_2, \phi(\zeta)]_T\) are such that \(d(q_i, p_i) \geq C_2\) then \([x_1, x_2]_T \subseteq [\psi(q_1), \psi(q_2)]_T\) and \(d(x_i, \psi(q_i)) \geq M = M(\psi)\).

Choose \(q_1 \in [\phi(\xi), p_1]_T\), \(q_2 \in [p_2, \phi(\zeta)]_T\) are such that \(d(q_i, p_i) = C_2\). Thus \([x_1, x_2]_T \subseteq [\psi(q_1), \psi(q_2)]_T\) and \(d(x_i, \psi(q_i)) \geq M\). Note that \(d(x_i, \psi(q_i)) \leq C_3 = C_3(\psi)\) since \(\psi\) is a quasi-isometry and \(d(x'_1, q_1) \leq C_2 + C_1\).

Thus \([x_1, x_2] \subseteq [\psi(q_1), \psi(q_2)]\) and \(d(x_i, \psi(q_i)) \geq M = M(\psi)\). Then by Proposition 5.4, applied to \(\psi\), we have

\[
\psi(Cyl_{T'}([q_1, q_2])) \subseteq Cyl_T([x_1, x_2])
\]

Applying \(\phi\), we obtain

\[
Cyl_{T'}([q_1, q_2]) \subseteq \phi(Cyl_T([x_1, x_2]))
\]

Note that by construction \([p_1, p_2] = [x'_1, x'_2] \cap [q_1, q_2]\). Moreover, \(d(q_i, x'_i) \leq C_1 + C_2\). Thus all the requirements of the proposition are satisfied, which completes the proof. \(\square\)

**Corollary 5.6.** Let \(\mu \in Curr(F)\) and let \(T_1, T_2 \in cv(F)\). Then

\[
h_{T_2}(\mu) > 0 \iff h_{T_2}(\mu) > 0.
\]

Proof. Suppose that \(h_{T_2}(\mu) > 0\). Hence there exists \(s > 0\), \(C > 0\) such that for every \(x, y \in T_2\), \(x \neq y\), we have

\[
\mu(Cyl_{T_2}([x, y])) \leq C \exp(-s d_{T_2}(x, y))
\]

Let \(\phi : T_1 \rightarrow T_2\) be a \(F\)-equivariant \((\lambda, \lambda)\)-quasi-isometry, where \(\lambda \geq 1\). Let \(M, C > 0\) be provided by Proposition 5.4. Let \(x_1, x_2 \in T_1\) be such that \(N := d_{T_1}(x_1, x_2) >
20\lambda^2 M. Let \( x_i = \phi(x_i) \in T_2, \ i = 1, 2 \). Thus \( d_{T_2}(x_1', x_2') \geq N/\lambda - \lambda \geq N/2\lambda \). Let 
\( y_1, y_2 \in [x_1', x_2']_{T_2} \) be such that \( d_{T_2}(x_1', y_1) = d_{T_2}(x_2', y_2) = M \). Thus 
\[
    d_{T_2}(y_1, y_2) \geq \frac{N}{2\lambda} - 2M \geq \frac{N}{3\lambda} = \frac{d_{T_1}(x_1, x_2)}{3\lambda}.
\]

By Proposition 5.4 we have:
\[
    \phi(Cyl_{T_1}([x_1, x_2])) \subseteq Cyl_{T_2}([y_1, y_2])
\]

Recall that under the identifications of \( \partial T_1 \) and \( \partial T_2 \) with \( \partial F \), for any \( S \subseteq \partial T_1 \) the set \( S \subseteq \partial T_2 \) and the set \( \phi(S) \subseteq \partial F \) determine the same subset \( S' \subseteq \partial F \).

Therefore 
\[
    \mu(Cyl_{T_1}([x_1, x_2])) \leq \mu(Cyl_{T_2}([y_1, y_2])) \leq C \exp(-s d_{T_2}(y_1, y_2)) \leq C \exp(-s \frac{d_{T_1}(x_1, x_2)}{3\lambda}).
\]

This implies that \( h_{T_1}(\mu) \geq \frac{s}{3\lambda} > 0 \), as required. \( \square \)

Corollary 5.6 implies that the following notion is well-defined:

**Definition 5.7** (Currents with exponential decay). Let \( \mu \in Curr(F) \). We say that \( \mu \) has exponential decay or decays exponentially fast if for some (equivalently, for any) \( T \in \text{cv}(F) \) we have \( h_T(\mu) > 0 \).

Similarly, we say that \( \mu \) has subexponential decay if for some (equivalently, for any) \( T \in \text{cv}(F) \) we have \( h_T(\mu) = 0 \).

**Definition 5.8.** Let \( \alpha : F \to \pi_1(\Gamma, p) \) be a simplicial chart and let \( \mathcal{L} \) be a metric graph structure on \( \Gamma \), so that \( (\alpha, \mathcal{L}) \) defines a point \( T \in \text{cv}(F) \). Let \( x \) be a vertex of \( T \), let \( \xi \in \partial F \) and let \( [x, \xi]_T \) be a geodesic ray from \( x \) to \( \xi \) in \( T \).

Put 
\[
    h_{T,x,\xi}(\mu) := \liminf_{d_T(y, z) \to \infty \atop [y, z]_T \in [x, \xi]_T} -\frac{\log \mu(Cyl_T([y, z]))}{d_T(y, z)}.
\]

Here \([y, z]_T \subseteq [x, \xi]_T\) means that \([y, z]\) is a subsegment of the ray \([x, \xi]_T\) respecting the orientation, that is, \( d_T(x, y) < d_T(x, z) \).

**Lemma 5.9.** Let \( \alpha, T, \mu, \xi \) be as in Definition 5.8. Then for any two vertices \( x_1, x_2 \) of \( T \) we have
\[
    h_{T,x_1,\xi}(\mu) = h_{T,x_2,\xi}(\mu)
\]

**Proof.** Let \( q \in T \) be such that \([x_1, \xi]_T \cap [x_2, \xi]_T = [q, \xi]_T\). It is enough to prove that \( h_{T,x_1,\xi}(\mu) = h_{T,q,\xi}(\mu) = h_{T,x_2,\xi}(\mu) \).

Therefore it suffices to consider the case where \( x_2 \in [x_1, \xi]_T \). Then it is obvious from the definition that
\[
    h_{T,x_1,\xi}(\mu) \leq h_{T,x_2,\xi}(\mu)
\]

We need to show that \( h_{T,x_2,\xi}(\mu) \leq h_{T,x_1,\xi}(\mu) \).

Let \([y_n, z_n]_T \subseteq [x_1, \xi]_T\) be such that \( \lim_{n \to \infty} d_T(y_n, z_n) = \infty \) and such that
\[
    h_{T,x_1,\xi}(\mu) := \liminf_{d_T(y, z) \to \infty \atop [y, z]_T \subseteq [x_1, \xi]_T} -\frac{\log \mu(Cyl_T([y, z]))}{d_T(y, z)} = \lim_{n \to \infty} -\frac{\log \mu(Cyl_T([y_n, z_n]))}{d_T(y_n, z_n)}.
\]

If for infinitely many \( n \) we have \([y_n, z_n] \subseteq [x_2, \xi]_T\), then
Notation 5.10. Let \( h \in \mathcal{H} \) and hence \( \text{Cyl}_n \) for all \( n \).

Thus we may assume that for every \( n \geq 1 \) we have \( d_T(x_1, y_n) \leq d_T(x_1, x_2) \).

Proposition 5.11. Let \( \alpha, T, \mu, \xi \) be as in Definition 5.8. Denote \( h_{T, \xi}(\mu) \) as required.

Then the following hold:

1. For any \( \xi \in \partial F \) we have \( h_T(\mu) \leq h_{T, \xi}(\mu) \).
2. There exists \( \xi \in \partial F \) such that \( h_T(\mu) = h_{T, \xi}(\mu) \).

Thus

\[
h_T(\mu) = \min_{\xi \in \partial F} h_{T, \xi}(\mu).
\]

Proof. Part (1) of the proposition is obvious. Part (2) of the proposition follows from the definitions of \( h_{T, \xi}(\mu) \) and of \( h_T(\mu) \) by the standard compactness argument.

Proposition 5.12. Let \( \mu \in \text{Curr}(F), \mu \neq 0 \). Then the function

\[
E_\mu : \text{cv}(F) \to \mathbb{R}, \ T \mapsto h_T(\mu)
\]

is continuous.

Proof. It suffices to check that that for every simplicial chart \( \alpha : F \to \pi_1(\Gamma, p) \) on \( F \) the restriction of \( E_\mu \) to the elementary chart \( V_n \subseteq \text{cv}(F) \) is continuous.

Let \( E^+ = E^+ \Gamma \cup E^- \Gamma \) be an orientation on \( \Gamma \), let \( m = #E^+ \Gamma \) and let \( E^+ \Gamma = \{ e_1, \ldots, e_m \} \). Recall that \( V_n \) consists of all points of the form \( (\alpha, \mathcal{L}) \), where \( \mathcal{L} \) is a nondegenerate semi-metric structure on \( \Gamma \).

Let \( T = (\alpha, \mathcal{L}) \in U(\alpha) \). Let \( \epsilon > 0 \) be arbitrary. Choose \( \epsilon_1 > 0 \) such that \( \epsilon_1 h_T(\mu) < \epsilon \). Then there exists a neighborhood \( \Omega \) of \( T \) in \( V_n \) with the following property. For any \( v \in \mathcal{P}(\Gamma) \) and for any \( T' = (\alpha', \mathcal{L}') \in \Omega \) we have

\[
(1 - \epsilon_1)\mathcal{L}'(v) \leq \mathcal{L}(v) \leq (1 + \epsilon_1)\mathcal{L}'(v).
\]
Therefore for any \( v \in \mathcal{P}(\Gamma) \) and any \( T' = (\alpha, \mathcal{L}') \in \Omega \) we have:

\[
- \log(v, \mu) (1 - \epsilon) \leq - \log(v, \mu) _\alpha (1 + \epsilon). 
\]

It follows that

\[
h_T(\mu)(1 - \epsilon) \leq h_T(\mu) \leq h_T(\mu)(1 + \epsilon)
\]

and hence

\[
|E_\mu(T) - E_\mu(T')| = |h_T(\mu) - h_T(\mu)| \leq h_\mu(T) \epsilon_1 \leq \epsilon.
\]

Thus \( E_\mu \) is continuous at the point \( T \) and, since \( T \in V_\alpha \) was arbitrary, \( E_\mu \) is continuous on \( V_\alpha \), as required.

**Remark 5.13.** Note that for a fixed \( T \in cv(F) \) the geometric entropy function \( h_T(\cdot) : Curr(F) \to \mathbb{R}, \mu \mapsto h_T(\mu) \) is not continuous. Indeed, for every rational current \( \lambda \eta \in Curr(F) \) we have \( h_T(\lambda \eta) = 0 \). Rational currents are dense in \( Curr(F) \) but there are many currents \( \mu \in Curr(F) \) with \( h_T(\mu) > 0 \). Thus \( h_T(\cdot) \) is indeed not a continuous function.

6. **TAME CURRENTS**

**Definition 6.1.** (Tame current with respect to a tree). Let \( \mu \in Curr(F) \) and \( T \in cv(F) \). We say that \( \mu \) is tame with respect to \( T \) or \( T \)-tame if for every \( M \geq 1 \) there is \( C = C(M) \geq 1 \) such that whenever \( a_1, b_1, a_2, b_2 \in T \) satisfy \( d(a_1, a_2) \leq M, d(b_1, b_2) \leq M, a_1 \neq b_1, a_2 \neq b_2 \) then

\[
\frac{1}{C} \mu(Cyl_T([a_2, b_2])) \leq \mu(Cyl_T([a_1, b_1])) \leq C \mu(Cyl_T([a_2, b_2])).
\]

We call \( C = C(M) \) the \( T \)-tameness constant corresponding to \( M \).

**Lemma 6.2.** Let \( T \in cv(F) \) and \( \mu \in Curr(F) \). Suppose that there is some \( N \geq 1 \) such that for every \( M \geq N \) there exists \( D = D(M) \geq 1 \) such that whenever \( [x, y] \subseteq [a, b] \) where \( x \neq y \) and \( d_T(a, x) = d_T(b, y) = M \) then

\[
\mu(Cyl_T([x, y])) \leq D \mu(Cyl_T([a, b])).
\]

Then \( \mu \) is \( T \)-tame.

**Proof.** Suppose that for every \( M \geq N \) there is \( D = D(M) \geq 1 \) as in Lemma 6.2.

We need to prove that \( \mu \) is \( T \)-tame. It is easy to see that it suffices to verify the conditions of Definition 6.1 for all sufficiently large \( M \).

Let \( M \geq N \) be arbitrary. Suppose now that \( [x, y] \subseteq [a, b] \) with \( d_T(a, x) = d_T(y, b) \leq M \) and \( x \neq y \). Choose a geodesic segment \([a', b']\) in \( T \) such that \( [a, b] \subseteq [a', b'] \) and such that \( d_T(a', x) = d_T(y, b') = M \). Then \( Cyl_T([a', b']) \subseteq Cyl_T([a, b]) \subseteq Cyl_T([x, y]). \) Hence by assumption on \( D = D(M) \) we have

\[
\mu(Cyl_T([a, b])) \leq \mu(Cyl_T([x, y])) \leq D \mu(Cyl_T([a', b'])) \leq D \mu(Cyl_T([a, b]))
\]

so that

\[
(\dagger) \quad \mu(Cyl_T([a, b])) \leq \mu(Cyl_T([x, y])) \leq D \mu(Cyl_T([a, b])).
\]

Thus (\dagger) holds whenever \( [x, y] \subseteq [a, b] \) with \( d_T(a, x) = d_T(y, b) \leq M \).

Suppose now that \( M \geq N \) and \( a_1, b_1, a_2, b_2 \in T \) satisfy \( d(a_1, a_2) \leq M, d(b_1, b_2) \leq M \). We may assume that \( d(a_1, b_1) \geq 3M \) since otherwise the requirements of Definition 6.1 are easily satisfied. (Indeed, if \( d(a_1, b_1) \leq 3M \), then the points \( a_1, a_2, b_1, b_2 \) lie in an \( F \)-translated of a fixed closed ball of radius \( 6M \), which is a
finite subtree of \( T \). Then \([a_1, b_1] \cap [a_2, b_2]\) is a non-degenerate geodesic segment. Put \([x, y] = [a_1, b_1] \cap [a_2, b_2]\).

Then
\[
d(x, a_1), d(x, a_2), d(y, b_1), d(y, b_2) \leq M
\]
and \([x, y] \subseteq [a_1, b_1], [x, y] \subseteq [a_2, b_2]\). Then by (1) we have
\[
\mu(CylT([a_1, b_1])) \leq \mu(CylT([x, y])) \leq D\mu(CylT([a_2, b_2]))
\]
and
\[
\mu(CylT([a_2, b_2])) \leq \mu(CylT([x, y])) \leq D\mu(CylT([a_1, b_1])).
\]
Therefore \( \mu \) is \( T \)-tame with as required. \( \square \)

**Proposition 6.3.** Let \( \mu \in Curr(F) \) and let \( T, T' \in cv(F) \). Then \( \mu \) is tame with respect to \( T \) if and only if \( \mu \) is tame with respect to \( T' \).

**Proof.** Suppose that \( \mu \) is tame with respect to \( T' \).

Let \( x \in T \) and \( x' \in T' \) be arbitrary vertices. Let \( \phi : T \to T' \) be an \( F \)-equivariant \((\lambda, \lambda)\)-quasi-isometry such that \( \phi(x) = x' \). Let \( M = M(\phi) \geq 1 \) be the constant provided by Proposition 5.4.

We need to prove that \( \mu \) is tame with respect to \( T \). By Lemma 6.2 it suffices to show that the conditions of Lemma 6.2 hold for \( \mu \).

Let \( M_0 \geq 1 \) be sufficiently big (to be specified later) and suppose \( s, t, a, b \in T \) are such that \([s, t]_T \subseteq [a, b]_T \) with \( d_T(s, a) = d_T(t, b) = M_0 \).

Let \( s', t' \in [\phi(a), \phi(b)]_{T'} \) be such that
\[
d_{T'}(\phi(s), s') = d_{T'}(\phi(s), [\phi(a), \phi(b)]_{T'}) \quad \text{and} \quad d_{T'}(\phi(t), t') = d_{T'}(\phi(t), [\phi(a), \phi(b)]_{T'}).
\]

Since \( \phi \) is a quasi-isometry and \( T, T' \) are Gromov-hyperbolic, we have \( d_{T'}(\phi(s), s'), d_{T'}(\phi(t), t') \leq C_1 \) where \( C_1 = C_1(\phi) > 0 \) is some constant. Note that \([\phi(a), \phi(b)] \cap [\phi(s), \phi(t)] = [s', t']\). We may assume that \( M_0 \) was chosen big enough so that
\[
d_{T'}(\phi(a), s'), d_{T'}(\phi(b), t') \geq M_1 = M_1(\psi)
\]
where \( M_1 = M_1(\psi) \) is the constant provided by Proposition 5.5.

Proposition 5.5 implies that there exist \( p_1, p_2, q_1, q_2 \in T' \) such that \([p_1, p_2]_{T'} = [q_1, q_2]_{T'} \cap [\phi(a), \phi(b)]_{T'} \) and such that
\[
Cyl_{T'}([q_1, q_2]_{T'}) \subseteq \phi(Cyl_T([a, b]_T))
\]
and such that \( d_{T'}(q_1, \phi(a)), d_{T'}(q_2, \phi(b)) \leq M_1 \). Thus \( d_{T'}(\phi(a), s'), d_{T'}(\phi(b), t') \geq M_1 = M_1(\psi) \) implies that \([s', t']_{T'} \subseteq [p_1, p_2]_{T'}\).

Let \( s'', t'' \in [\phi(a), \phi(b)]_{T'} \) be such that \( d_{T'}(s', s'') = d_{T'}(t', t'') = M \).

Thus \( s'', t'' \in [\phi(a), \phi(b)]_{T'} \) and
\[
M \leq d_{T'}(\phi(s), s'') \leq M + C_1, \quad M \leq d_{T'}(\phi(t), t'') \leq M + C_1.
\]

Then by Proposition 5.4
\[
\phi(Cyl_{T'}([s, t]_{T'})) \subseteq Cyl_{T'}([s'', t'']_{T'}),
\]
and hence
\[
\mu(Cyl_{T'}([s, t]_{T'})) \leq \mu(Cyl_{T'}([s'', t'']_{T'})).
\]

Note that since \( \phi \) is a \((\lambda, \lambda)\)-quasi-isometry and \( d_T(a, s) = M_0, d_T(b, t) = M_0 \) then
\[
d_{T'}(\phi(a), \phi(s), d_{T'}(\phi(b), \phi(t)) \leq \lambda M_0 + \lambda.
\]
Since $d_T(\phi(s), s')$, $d_T(\phi(t), t') \leq C_1$, it follows that
\[ d_T(\phi(a), s'), d_T(\phi(b), t') \leq \lambda M_0 + \lambda + C_1. \]
Since $d_T(s', s'') = d_T(t', t'') = M$, we have
\[ d(\phi(a), s''), d(\phi(b), t'') \leq M + \lambda M_0 + \lambda + C_1. \]
Since $p_1 \in [\phi(a), s'']_T$, $p_2 \in [t'', \phi(b)]_T$, and $d_T(q_1, \phi(a)), d_T(q_2, \phi(b)) \leq M_1$, we get
\[ d_T(s'', q_1), d_T(t'', q_2) \leq M_1 + C + \lambda M_0 + \lambda + C_1. \]
Put $M_2 = M_1 + M + \lambda M_0 + \lambda + C_1$. Since $\mu$ is $T'$-tame,
\[ \mu(Cyl_T([s'', t'']) \leq C_2 \mu(Cyl_T([q_1, q_2]_T)), \]
where $C_2 = C(M_2)$ is the $T'$-tameness constant for $\mu$ corresponding to $M_2$.
Recall that $Cyl_T([q_1, q_2]_T) \subseteq \phi(Cyl_T([a, b]_T))$ and therefore
\[ \mu(Cyl_T([q_1, q_2]_T)) \leq \mu(Cyl_T([a, b]_T)). \]
Thus we have
\[ \mu(Cyl_T([s, t]_T)) \leq \mu(Cyl_T([s'', t'']_T)) \leq C_2 \mu(Cyl_T([q_1, q_2]_T)) \leq \mu(Cyl_T([a, b]_T)). \]
Hence by Lemma 6.2 $\mu$ is $T$-tame, as required.

Proposition 6.3 implies that the following notion is well-defined and does not depend on the choice of $T \in cv(F)$:

**Definition 6.4** (Tame current). Let $\mu \in Curr(F)$. We say that $\mu$ is tame if for some (equivalently, for any) $T \in cv(F)$ the current $\mu$ is $T$-tame.

### 7. The geometric entropy function on the Outer Space

**Theorem 7.1.** Let $T \in cv(F)$ and let $\mu \in Curr(F)$ be a tame current. Then for any $T' \in cv(F)$ we have:
\[ h_{T'}(\mu) \geq h_T(\mu) \inf_{f \in F^+} \frac{|f|_T}{|f|_{T'}}. \]

**Proof.** Put $h = h_T(\mu)$.

Let $x \in T$ and $x' \in T'$ be arbitrary vertices. Let $\phi : T' \to T$ be an $F$-equivariant quasi-isometry such that $\phi(x') = x$. Thus $\phi(gx') = gx$ for every $g \in F$. Let $M \geq 1$ be provided by Proposition 5.4.

Let $a_n, b_n \in VT'$ be such that $\lim_{n \to \infty} d_{T'}(a_n, b_n) = \infty$ and such that
\[ h_{T'}(\mu) = \lim_{n \to \infty} -\log \mu(Cyl_{T'}([a_n, b_n])). \]

Note that there exists some constant $M' = M'(T') \geq 1$ such that for any reduced edge-path $v$ in $T'/F$ there exists a cyclically reduced closed edge-path $\hat{v}$ in $T'/F$ containing $v$ as a subpath and such that $|\mathcal{L}(v) - \mathcal{L}(\hat{v})| \leq M'$, where $\mathcal{L}$ is the metric graph structure on $T'/F$ induced from the metric on $T$.

Then there exists $h_n, g_n \in F$ such that $d_{T'}(a_n, h_n x') \leq M'$, $d_{T'}(b_n, g_n x') \leq M'$ and such that $[h_n x', g_n x']_T$ projects to a closed cyclically reduced path in $T'/F$. 

After translating \([a_n, b_n]\) by \(h_n^{-1}\), we may assume that \(h_n = 1\). Thus \([x', g_n x']_{T'}\) is contained in the axis if \(g_n\) and \(d_{T'}(x', g_n x') = \|g_n\|_{T'}\). Note that \(\lim_{n \to \infty} d_{T'}(a_n, b_n) = \infty\) implies \(\lim_{n \to \infty} \|g_n\|_{T'} = \infty\).

Since \(\mu\) is tame, we have
\[
\mu(Cyl_{T'}([a_n, b_n])) \leq C_1 \mu(Cyl_{T'}([x', g_n x'])).
\]
Therefore
\[
\frac{- \log \mu(Cyl_{T'}([a_n, b_n]))}{d_{T'}(a_n, b_n)} \geq \frac{- \log \mu(Cyl_{T'}([x', g_n x'])) - \log C_1}{d_{T'}(x', g_n x') + 2M'}.
\]
and
\[
h_{T'}(\mu) \geq \liminf_{n \to \infty} \frac{- \log \mu(Cyl_{T'}([x', g_n x']))}{d_{T'}(x', g_n x')}.
\]

Note that \(\phi(x') = x\) and \(\phi(g_n x') = g_n x\). Moreover, since \(d_{T'}(x', g_n x') = \|g_n\|_{T'}\) and \(\phi\) is a quasi-isometry, there is some \(C_3 > 0\) independent of \(n\) such that for every \(n \geq 1\)
\[
|d_T(x, g_n x) - \|g_n\|_{T'}| \leq C_3.
\]
Also, we have \(\lim_{n \to \infty} \|g_n\|_{T'} = \infty\). Let \([y_n, z_n] \subseteq [x, g_n x]\) be such that \(d_T(x, y_n) = d_T(g_n x, z_n) = M\).

By proposition 5.4
\[
\phi(Cyl_{T'}([x', g_n x'])) \subseteq Cyl_{T'}([y_n, z_n])
\]
and hence
\[
\mu(Cyl_{T'}([x', g_n x'])) \leq \mu(Cyl_{T'}([y_n, z_n])) \leq C_2 \mu(Cyl_{T'}([x, g_n x])),
\]
where the last inequality holds since \(\mu\) is tame.

Thus
\[
\frac{- \log \mu(Cyl_{T'}([x', g_n x']))}{d_{T'}(x', g_n x')} \geq \frac{- \log \mu(Cyl_{T'}([x, g_n x])) - \log C_2}{d_{T'}(x', g_n x')} = \frac{- \log \mu(Cyl_{T'}([x, g_n x])) - \log C_2}{d_{T'}(x', g_n x')} \frac{d_{T'}(x, g_n x)}{d_{T'}(x', g_n x')} = \frac{- \log \mu(Cyl_{T'}([x, g_n x])) - \log C_2}{d_{T'}(x, g_n x)} \frac{d_{T'}(x', g_n x')}{d_{T'}(x, g_n x)} \geq \frac{- \log \mu(Cyl_{T'}([x, g_n x])) - \log C_2}{\|g_n\|_{T'}} \frac{\|g_n\|_{T'} - C_3}{\|g_n\|_{T'}} \geq \frac{- \log \mu(Cyl_{T'}([x, g_n x])) - \log C_2}{\|g_n\|_{T'}} \frac{\|g_n\|_{T'} - C_3}{\|g_n\|_{T'}} \geq \frac{- \log \mu(Cyl_{T'}([x', g_n x']))}{\|g_n\|_{T'}} \frac{\|g_n\|_{T'} - C_3}{\|g_n\|_{T'}} \frac{\|g\|_{T'}}{\|g\|_{T'}} \geq h_{T'}(\mu) \inf_{g \in F} \frac{\|g\|_{T'}}{\|g\|_{T'}},
\]
as required. \(\square\)
The following statement is an immediate corollary of the explicit formula for the Patterson-Sullivan current in the Outer Space context obtained by Kapovich and Nagnibeda in [35] (see Proposition 5.3 of [35]):

**Proposition 7.2.** Let \( T \in cv(F) \) and let \( h = h(T) \) be the critical exponent of \( T \). Let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current corresponding to \( T \). Then there exist constants \( C_1 > C_2 > 0 \) such that for any distinct vertices \( x \) and \( y \) of \( T \) we have

\[
C_2 \exp(-h \, d_T(x, y)) \leq \mu_T(Cyl_T([x, y]_T)) \leq C_1 \exp(-h \, d_T(x, y)).
\]

Together with the definitions of geometric entropy and of tameness, Proposition 7.2 immediately implies:

**Corollary 7.3.** Let \( T \in cv(F) \) and let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current corresponding to \( T \). Let \( h = h(T) \) be the critical exponent of \( T \).

Then \( \mu_T \) is tame and \( h_T(\mu_T) = h(T) \).

**Proposition 7.4.** Let \( T \in cv(F) \) and let \( h = h(T) \) be the critical exponent of \( T \). Let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current corresponding to \( T \).

Let \( T' \in cv(F) \). Then for any \( g \in F^\times \) we have

\[
h_T(\mu) \leq h \frac{||g||_T}{||g||_{T'}}
\]

and therefore

\[
h_{T'}(\mu) \leq h \inf_{f \in F^\times} \frac{||f||_T}{||f||_{T'}}.
\]

**Proof.** By Proposition 7.2 there exists \( C > 0 \) such that for any distinct vertices \( x, y \in T \) we have

\[
\mu(\text{Cyl}_T([x, y])) \geq C \exp(-h d_T(x, y)).
\]

Note that for any \( x \in T, x' \in T' \) we have

\[
||g||_T = \lim_{n \to \infty} \frac{d_T(x, g^n x)}{n}, \quad ||g||_{T'} = \lim_{n \to \infty} \frac{d_{T'}(x', g^n x')}{n}
\]

and hence

\[
\lim_{n \to \infty} \frac{d_T(x, g^n x)}{d_{T'}(x', g^n x')} = \frac{||g||_T}{||g||_{T'}}.
\]

Let \( x \in T \) and \( x' \in T' \) be arbitrary vertices. Let \( \phi : T \to T' \) be an \( F \)-equivariant quasi-isometry such that \( \phi(x) = x' \). Thus \( \phi(g^n x) = g^n x' \) for every \( n \in \mathbb{Z} \). Let \( M \geq 1 \) be provided by Proposition 5.4. For \( n \to \infty \) let \( [y_n, z_n] \subseteq [x', g^n x'] \) be such that \( d(x', y_n) = d(z_n, g^n x') = M \). Then by Proposition 5.4 we have

\[
\phi(\text{Cyl}_T([x, g^n x])) \subseteq \text{Cyl}_{T'}([y_n, z_n]).
\]

Hence

\[
\mu(\text{Cyl}_{T'}[y_n, z_n]) \geq \mu(\text{Cyl}_T[x, g^n x]) \geq C \exp(-h d_T(x, g^n x))
\]

and so

\[
- \log \left( \frac{\mu(\text{Cyl}_{T'}[y_n, z_n])}{\mu(\text{Cyl}_T[x, g^n x])} \right) \leq \frac{h d_T(x, g^n x) - \log C}{d_{T'}(y_n, z_n)} \leq \frac{h d_T(x, g^n x) - \log C}{d_{T'}(x', g^n x') - 2M}.
\]

Hence

\[
h_{T'}(\mu) \leq \liminf_{n \to \infty} \frac{h d_T(x, g^n x) - \log C}{d_{T'}(x', g^n x') - 2M} = h \frac{||g||_x}{||g||_{T'}}
\]

as required. \( \square \)
Since Patterson-Sullivan currents are tame, Proposition 7.4 and Theorem 7.1 imply:

**Theorem 7.5.** Let \( T \in cv(F) \) and let \( h = h(T) \) be the critical exponent of \( T \). Let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current for \( T \). Let \( T' \in cv(F) \). Then

\[
h_{T'}(\mu_T) = h(T) \inf_{f \in F^*} \frac{||f||_T}{||f||_{T'}}.
\]

The following useful result is due to Tad White [53]:

**Proposition 7.6.** Let \( T_1, T_2 \in CV(F) \) be such that \( T_1 \neq T_2 \). Then there exist nontrivial \( g_1, g_2 \in F \) such that

\[
||g_1||_{T_1} < ||g_1||_{T_2} \quad \text{and} \quad ||g_2||_{T_2} < ||g_2||_{T_1}.
\]

**Corollary 7.7.** Let \( T \in CV(F) \) and let \( h = h(T) \) be the critical exponent of \( T \). Let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current for \( T \). Let \( T' \in CV(F) \) be such that \( T' \neq T \). Then

\[
h_{T'}(\mu_T) < h_T(\mu_T) = h(T).
\]

Thus

\[
h(T) = h_T(\mu_T) = \max_{T' \in CV(T)} h_{T'}(\mu_T)
\]

and this maximum is strict.

**Proof.** Since \( T' \neq T \), Proposition 7.6 implies that there exists some \( g \in F \) such that \( ||g||_T < ||g||_{T'} \). Therefore by Theorem 7.5

\[
h_{T'}(\mu_T) = h(T) \frac{||g||_T}{||g||_{T'}} < h(T),
\]

as required. \( \square \)

**Corollary 7.8.** Let \( T \in CV(F) \) and let \( \mu_T \in \text{Curr}(F) \) be a Patterson-Sullivan current for \( T \). Then

\[
\inf_{T' \in CV(F)} h_{T'}(\mu_T) = 0.
\]

**Proof.** Recall that \( F \) is a free group of rank \( k \geq 2 \). Let \( A \) be a free basis of \( F \) and let \( T_A \) be the Cayley graph of \( F \) with respect to \( A \), where every edge has length \( 1/k \). Thus \( T_A \in CV(F) \). Let \( a \in A \). There exists a sequence \( \phi_n \in \text{Out}(F) \) such that \( \lim_{n \to \infty} ||\phi_n a||_A = \infty \) and hence \( \lim_{n \to \infty} ||\phi_n a||_{T_A} = \frac{1}{k} \lim_{n \to \infty} ||\phi_n a||_A = \infty \).

Put \( T_n = \phi_n^{-1}T_A \). Thus \( T_n \in CV(F) \) and

\[
||a||_{T_n} = ||a||_{\phi_n^{-1}T_A} = ||\phi_n a||_{T_A} \xrightarrow{n \to \infty} \infty.
\]

Therefore by Theorem 7.5 we have

\[
h_{T_n}(\mu) \leq h(T) \frac{||a||_{T}}{||a||_{T_n}} \xrightarrow{n \to \infty} 0.
\]

Hence

\[
\inf_{T' \in CV(F)} h_{T'}(\mu) = 0,
\]

as required. \( \square \)
8. The maximal geometric entropy problem for a fixed tree

Recall that, as observed in Remark 5.13, the function $h_T(\cdot): Curr(F) \to \mathbb{R}, \mu \mapsto h_T(\mu)$ is not continuous. Nevertheless, it turns out that it is possible to find the maximal value of $h_T(\cdot)$ on $Curr(F) - \{0\}$.

Recall that if $(X, d)$ is a metric space and $\nu$ is a measure on $X$, then the Hausdorff dimension $HD_X(\nu)$ of $\nu$ with respect to $X$ defined as

$$HD_X(\nu) := \inf\{HD(S): S \subseteq X \text{ such that } \nu(X - S) = 0\}.$$

Thus $HD_X(\nu)$ is the smallest Hausdorff dimension of subsets of $X$ of full $\nu$-measure. Note that this obviously implies that $HD_X(\mu) \leq HD(X)$.

**Definition 8.1.** Let $T \in cv(F)$. For any point $x \in \partial T$ and $y \in T, y \neq x$ let $Cyl_T([x, y]) \subseteq \partial T$ denote the one-sided cylinder, that is the set of all $\xi \in \partial T$ such that $[x, \xi]$ has initial segment $[x, y]$.

For a current $\mu \in Curr(F)$ and $x \in T$ denote by $\mu_x$ the measure on $\partial F$ defined by:

$$\mu_x(Cyl_T([x, y])) := \mu(Cyl_T([x, y]))$$

for every $y \in T, y \neq x$. We say that $\mu_x$ is the measure on $\partial F$ corresponding to $\mu$ and $x$.

Note that if $\mu \in Curr(F), \mu \neq 0$ then there is $x \in T$ such that $\mu_x \neq 0$. If $\mu_T$ is a Patterson-Sullivan current corresponding to $T$ then $\mu_x$ is a Patterson-Sullivan measure on $\partial F$ corresponding to $T$ (see [35]).

Let $T \in cv(F)$ be arbitrary. Recall that if $x \in T, \xi, \zeta \in \partial T$, we denote by $(\xi|\zeta)_x$ the distance $d_T(x, y)$ where $[x, \xi] \cap [x, \zeta] = [x, y]$. Let $x \in T$ be a base-point. The boundary $\partial T$ is metrized by setting $d_T(\xi, \zeta) = \exp(- (\xi|\zeta)_x)$ for $\xi, \zeta \in \partial T$. It is well-known (see, for example, [14, 35]) that $HD(\partial T, d_x) = h(T)$.

**Theorem 8.2.** Let $\mu \in Curr(F), \mu \neq 0$ and let $x \in T$ be such that $\mu_x \neq 0$. Then

$$h_T(\mu) \leq HD_{\partial T}(\mu_x) \leq h(T).$$

**Proof.** As observed by Kaimanovich in [26], for the case where of the boundary of a metric tree, one can give a more explicit formula for the Hasudorff dimension of a measure. Namely, let $T \in cv(F)$ and let $x \in T$ be a base-point. With endow $\partial T$ with the metric $d_x$ as above. Then for a measure $\nu$ on $\partial T$ we have (see formula (1.3.3) in [26]):

$$\text{HD}_{\partial T}(\nu) = \text{ess sup}_{\xi \in \partial T} \liminf_{k \to \infty} \frac{-\log \nu(B_x(\xi, k))}{k} \tag{1}$$

Here $B_x(\xi, k)$ is the set of all $\zeta \in \partial T$ such that $(\xi|\zeta)_x \geq k$, that is $B_x(\xi, k) = Cyl_T([x, y])$ where $[x, y]$ is the initial segment of $[x, \xi]$ of length $k$. The essential supremum in (1) is taken with respect to $\nu$.

Applied to $\mu_x$, formula (1) yields:
Proposition 8.3. Let $T, T' \in \text{cv}(F)$ be such that $h := h(T) = h(T')$. Let $\mu_T$ be a Patterson-Sullivan current corresponding to $T$ and suppose that $h_T(\mu_T) = h$.

Then $[T] = [T']$.

Proof. Let $\mu_{T'} \in \text{Curr}(F)$ be a Patterson-Sullivan current corresponding to $T'$. We will first show that $\mu_T$ is absolutely continuous with respect to $\mu_{T'}$.

Let $\phi : T \to T'$ be an $F$-equivariant $(\lambda, \lambda)$-quasi-isometry, where $\lambda \geq 1$.

By Proposition 7.2 there is a constant $C \geq 1$ such that for any $x, y \in T$ with $d_T(x, y) \geq 1$

$$\frac{1}{C} \exp(-hd_T(x, y)) \leq \mu_T Cyl_T([x, y]) \leq C \exp(-hd_T(x, y)).$$

Let $x', y' \in \phi(T')$ be arbitrary such that $d_T(x', y') \geq \lambda^2 + \lambda$. Let $x, y \in T$ be such that $x' = \phi(x)$, $y' = \phi(y)$. Since $\mu_T$ is tame, Proposition 5.4 and Proposition 5.5 imply that there is some constant $C_1 \geq 1$ such that

$$\frac{1}{C_1} \mu_T Cyl_{T'}([x', y']) \leq \mu_T Cyl_T([x, y]) \leq C_1 \mu_T Cyl_{T'}([x', y']).$$

Thus

$$\mu_T Cyl_{T'}([x', y']) \geq \frac{1}{C_1} \mu_T Cyl_T([x, y]) \geq \frac{1}{C_1 C} \exp(-hd_T(x, y)).$$

On the other hand, since by assumption $h_T(\mu_T) = h$, it follows that for any $\epsilon > 0$ there exists $C_2 = C_2(\epsilon) \geq 1$ such that

$$\mu_T Cyl_{T'}([x', y']) \leq C_2 \exp(-(h - \epsilon)d_T(x, y)).$$

Thus

$$\frac{1}{C_1 C} \exp(-hd_T(x, y)) \leq C_2 \exp(-(h - \epsilon)d_T(x, y)).$$

Hence

$$hd_T(x, y) \geq (h - \epsilon)d_T(x', y') - \log(C_2 C_1).$$

and so

$$\frac{h}{h - \epsilon} \geq \limsup_{d_T(x, y) \to \infty} \frac{d_T(x', y')}{d_T(x, y)}.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\limsup_{d_T(x, y) \to \infty} \frac{d_T(x', y')}{d_T(x, y)} \leq 1.$$
Therefore
\[ \sup_{g \in F^x} \frac{||g||_T}{||g||_{T'}} \leq 1. \]

By Lemma 3.9 this implies that there is a constant \( C_3 = C_3 \geq 1 \) such that
\[ d_T(x', y') \leq d_T(x, y) + C_3. \]

Then
\[ \mu_T(Cyl_T'[x', y']) \leq \frac{1}{C_1 C} \exp(-h d_T(x, y)) \leq \frac{C'}{C_1 C} \exp(h C_3) \exp(-h d_T(x', y')) \leq \frac{C'}{C_1 C} \mu_T'(Cyl_T'[x', y']). \]

The above inequality holds for any \( x', y' \in \phi(T) \) with \( d_T(x', y') \geq \lambda^2 + \lambda \). Since \( \phi \) is a quasi-isometry and \( \mu_T \) is tame, it follows that there exists a constant \( C' \geq 1 \) such that for any \( x', y' \in T' \) with \( d_T(x', y') \geq 1 \) we have
\[ \mu_T(Cyl_T'[x', y']) \leq C' \mu_T'(Cyl_T'[x', y']). \]

Hence \( \mu_T \) is absolutely continuous with respect to \( \mu_T' \).

A result of Furman [20] now implies that the translation length functions \( ||.||_T \) and \( ||.||_{T'} \) are scalar multiples of each other, that is, \([T] = [T']\), as required.

**Corollary 8.4.** Let \( T_1, T_2 \in cv(F) \) be such that \( [T_1] \neq [T_2] \). Let \( \mu_{T_2} \) be a Patterson-Sullivan current for \( T_2 \).

Then \( h_{T_1}(\mu_{T_2}) < h(T_1) \).

**Proof.** After replacing \( T_2 \) by a scalar multiple of \( T_2 \) we may assume that \( h(T_1) = h(T_2) \). Note that the projective Patterson-Sullivan current depends only on the projective class of an element of \( cv(F) \), so that this replacement does not change \( \mu_{T_2} \). Now the statement of the corollary follows immediately from Theorem 8.2 and Proposition 8.3.

**Corollary 8.5.** Let \( T, T' \in cv(F) \). Then:

1. \[ \inf_{g \in F^x} \frac{||g||_T}{||g||_{T'}} \leq \frac{h(T')}{h(T)} \leq \sup_{g \in F^x} \frac{||g||_T}{||g||_{T'}}. \]

2. Suppose that \([T] \neq [T']\). Then
\[ \inf_{g \in F^x} \frac{||g||_T}{||g||_{T'}} < \frac{h(T')}{h(T)} < \sup_{g \in F^x} \frac{||g||_T}{||g||_{T'}}. \]

**Proof.**

(1) Let \( \mu_T \) be a Patterson-Sullivan current corresponding to \( T \). By Theorem 7.5 and Theorem 8.2 we have
\[ h(T) \inf_{g \in F^x} \frac{||g||_T}{||g||_{T'}} = h_{T'}(\mu_T) \leq h(T') \]
and hence
\[ \inf_{g \in F^x} \frac{||g||_T}{||g||_{T'}} \leq \frac{h(T')}{h(T)}. \]
A symmetric argument shows that
\[ \inf_{g \in F} \frac{||g||_T}{||g||_{T'}} \leq \frac{h(T)}{h(T')} \]
Clearly,
\[ \inf_{g \in F} \frac{||g||_T}{||g||_{T'}} = \frac{1}{\sup_{g \in F} \frac{||g||_T}{||g||_{T'}}} \]
and hence
\[ \sup_{g \in F} \frac{||g||_T}{||g||_{T'}} \geq \frac{h(T')}{h(T)} \]
as required.

The proof of part (2) is exactly the same, but Corollary 8.4 implies that all the inequalities involved are now strict. □

References

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