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ON MATHEMATICAL CONTRIBUTIONS OF PAUL E. SCHUPP

ILYA KAPOVICH

1. Biographical data

Paul Eugene Schupp was born on March 12, 1937 in Cleveland, Ohio. He obtained a Bachelor's degree at Case Western Reserve University in 1959. During his undergraduate studies there in 1955–1959, Paul became interested in mathematics. In 1959, he became a Ph.D. student in Mathematics at the University of Michigan. He completed his Ph.D. in 1966, under the direction of Roger Lyndon.

After graduating from Michigan, Paul was a visiting professor at the University of Wisconsin–Madison for the 1966–1967 academic year. He then spent the 1967–1968 academic year at UIUC at the invitation of William Boone, as a visitor for the special year in Combinatorial Group Theory. At the conclusion of that special year, Paul became an Assistant Professor at UIUC and he remained a faculty member ever since. Paul was promoted to Associate Professor in 1971 and to Professor in 1975.

While a faculty member at UIUC, Paul Schupp has held visiting appointments at the Courant Institute (1969–1970), University of Singapore (January–April 1982), University of London (April–September 1982), USSR National Academy of Sciences in Moscow (September–December 1982), University of Bordeaux (1984 and 1996), University of Paris—VII (1984–1992), Université Marne-la-Vall'e, June 1999 and June 1997, and others.

He was awarded the John Simon Guggenheim Fellowship for the 1977–1978 academic year.

During his years at UIUC Paul was the thesis advisor of twelve Ph.D. students, including Leo Comerford (1973), Judith Seymour (1974), Richard Hurwitz (1974), David Jackson (1978), Robert Haring-Smith (1981), Claude Anderson (1981), Peter Lindsay (1984), Maximiliano Garzon (1984), David

Peifer (1992), Karin Johnsgard (1993), Paul Gies (1995) and Gareth Rohde (1997).

Paul Schupp retired from UIUC at the end of the 2007–2008 academic year and he has been appointed a Professor Emeritus in the Department of Mathematics of the University of Illinois at Urbana–Champaign.

2. Mathematical contributions

Paul Schupp is a group theorist by "mathematical birth": his 1966 Ph.D. thesis at the University of Michigan, under the direction of Roger Lyndon, was titled *On Dehn's Algorithm and the Conjugacy Problem*. Most of Paul's subsequent mathematical research has been in the subject of Group Theory. However, he has also made fundamental contributions to Computer Science and Complexity Theory, some of which are discussed below. In fact, from 1984 to 1992 he was a long-term visitor at the Computer Science Department of the University of Paris—VII.

Paul's work in Group Theory has had a fundamental impact on the subject and on the transformation of Combinatorial Group Theory into Geometric Group Theory, which took place in late 1980s–early 1990s.

Paul's early research, starting with his Ph.D. thesis, concerned the development and applications of small cancellation theory. The ideas behind small cancellation theory originated in the work of Tartakowski in the 1940s, but the subject was only put on the firm mathematical footing by Greendlinger in 1960s [6], [7], [8]. The work of Schupp and Lyndon, in 1960s and 1970s, pushed the topic further, by developing a nonmetric version of small cancellation theory, as well as versions of small cancellation theory over amalgamated products and HNN-extensions. The treatment of small cancellation theory in the Lyndon and Schupp book [22] remains the modern standard for the subject. Small cancellation theory provided the first version of negative curvature type conditions for group presentations. For this reason, small cancellation theory, and the work of Schupp in particular [36], [37], [38], [39], [40], was an important precursor of the notion of a word-hyperbolic group, introduced by Gromov in his seminal 1987 monograph "Hyperbolic groups" [9]. The advent of the theory of word-hyperbolic groups, in turn, signalled the birth of Geometric Group Theory as a distinct subject (even though geometric ideas did play an important role in the study of infinite groups essentially since the work of Dehn in 1910s). In his 1973 survey on small cancellation theory [40], Schupp, rather prophetically, asked:

"What 'is' a small cancellation group? What is desired here is a geometric characterization of small cancellation groups. [...] The geometric approach to small cancellation theory suggests that there should be a characterization of small cancellation groups by means of 'natural' geometric properties of their

Cayley diagrams, or in terms of their possible action on other complexes. Such a characterization would bring us full circle back to Dehn."

Of course, this is exactly what the notion of a word-hyperbolic group accomplished when it was defined by Gromov some 15 years later. Small cancellation theory has seen renewed relevance and applications in recent years, particularly in the work of Gromov on random groups [10] (where the "small cancellation with respect to a graph" condition was introduced and explored) and in various applications of small cancellation type quotients of hyperbolic and relatively hyperbolic groups [32].

Schupp's 1972 paper with Appel [2] is a perfect example of an early Geometric Group Theory result: they used topological features of prime alternating knots to derive certain small cancellation (essentially nonpositive curvature) properties of Dehn presentations of the corresponding knot groups, in order to conclude that these groups have solvable conjugacy problem. This kind of reasoning would later become a hallmark of Geometric Group Theory. A 1983 paper of Appel and Schupp [3], published in Inventiones, obtained important applications of small cancellation methods to the study of algebraic and algorithmic properties of Artin and Coxeter groups.

In the subject of Group Theory the name of Paul Schupp is most closely associated with his seminal 1977 book, joint with Roger Lyndon, *Combinatorial Group Theory* [22]. The book provided a comprehensive account of the subject of Combinatorial Group Theory, starting with the work of Dehn in the 1910s and to late 1970s. It contained a remarkable amount of material, including in-depth proofs (for both classic and modern "state of the art" results) as well as survey-type chapters.

Lyndon and Schupp, as the book quickly became known, soon came to be regarded as the definitive work in the area of Combinatorial Group Theory. Even today, after much progress over the last 30 years, and the transformation of Combinatorial Group Theory into Geometric Group Theory, that took place in late 1980s to mid 1990s, Lyndon and Schupp remains a "must read", and indeed a "must own", for the researchers and students in the area. Quite deservedly, the book was reprinted by Springer in 2001 in its *Classics in Mathematics* series.

Much of Paul's work in 1980s and 1990s turned towards Computer Science and towards exploring interactions between Computer Science and Complexity theory on one side and Group Theory on the other side. Again, this theme was later to become central in Geometric Group Theory.

His most famous result in this direction is a theorem, obtained in a joint 1983 paper with David Muller [24], which states that a finitely presented group has contex-free word problem (i.e., where the set of all words over a finite generating set of the group, representing the identity element in the group, is a context-free language) if and only if this group is virtually free. This result provides a far-reaching generalization of the classic theorem of Anisimov [1] that a finitely generated group is finite if and only if its word problem (over any finite generating set) is a regular language. The Muller–Schupp theorem is much more difficult and requires considerably more sophisticated tools and ideas. The hardest direction is to show that if a finitely presented group G has contex-free word problem then the group is virtually free. Here, Muller and Schupp use innovative language-theoretic and geometric ideas to show that such a group G has infinitely many ends. By Stallings' theorem about ends of groups [42], it follows that G splits nontrivially, as an amalgamated free product or an HNN-extension, over a finite group. One then shows that the base group(s) of such a splitting is(are) again finitely presented with contex-free word problem. Iterating this process, and using Dunwoody's result about accessibility of finitely presented groups [5], it follows that G is the fundamental group of a finite graph of finite groups and hence it is virtually free.[1] This paper of Muller and Schupp signalled the start[2] of their fruitful and active collaboration, sometimes involving additional coauthors, that lasted through 1980s and 1990s [24], [25], [26], [27], [28], [29], [35]. The main themes of this collaboration were exploring infinite automata, monadic logic, and the theory of ends, both for graphs and groups. Among this joint work of Muller and Schupp, another paper that deserves special mention here is their 1985 paper [25]. In this article they obtain a remarkable generalization of the classic 1969 result of Rabin [33] that the full monadic theory of the binary tree is decidable. Muller and Schupp prove that the monadic second order logic of any finitely generated contex-free graph is decidable. Being finitely generated for a graph is a mild generalization of having bounded vertex degree; in particular Cayley graphs of finitely generated groups are finitely generated as graphs. The notion of a context-free graph is motivated by the Muller and Schupp previous paper [24], discussed above, where connections between push-down automata and ends of graphs were first explored. Roughly speaking, a finitely generated graph Γ is *context free* if there are only finitely many isomorphism types of infinite graphs that one obtains as infinite connected components of complements to finite subgraphs in Γ. In particular, full transition graphs of push-down automata and Cayley graphs of virtually free groups are context free. Another paper from the Muller–Schupp collaboration sequence, deserving papericular mention, is their 1987 paper [26], which introduced and explored a new natural model of alternating automata on infinite trees.

A 1998 paper of Ivanov and Schupp [12] revisits the topic of small cancellation theory in the modern geometric group theory context. They study word-hyperbolicity for a class of one-relator groups, via exploring "flat" small

[1] In fact, in [24] Muller and Schupp prove that a finitely generated group is virtually free if and only if it is accessible and has contex-free word problem. By a subsequent result of Dunwoody that all finitely presented groups are accessible, it follows that the Muller and Schupp theorem covers all finitely presented groups.

[2] It was preceded by a short joint research announcement [23].

cancellation conditions that lie on the boundary between negative and non-positive curvature.

Another important direction in the work of Schupp is the use of computer experiments in mathematical research. In his 1998 paper [41], he explores a long-standing conjecture that every Cayley graph of a finite group admits a Hamiltonian cycle. Using an ingenious combination of mathemtical intuition and computer experimentation, Schupp finds the presence of Hamiltonian cycles in a family of finite quotient of the modular group $PSL(2,\mathbb{Z})$. Paul refers to the patterns, discovered in [41], predicting the specific structure of Hamiltonian cycles in that context as "mystic numerology".

Much of Paul's research in the last ten years concerned the study of generic properties of groups and algorithms [13], [14], [15], [16], [17], [18], [19], [20], [21]. A joint 2003 paper of Schupp, with Kapovich, Miasnikov and Schpilrain [14], introduced the notion of *generic-case complexity* and explored this notion in the context of group-theoretic algorithms. Informally, generic-case complexity captures the complexity of an algorithm on "most" (in some natural probabilistic sense) inputs, while ignoring its behavior on a small ("negligible") set of inputs. Thus, generic-case complexity has substantially different features from average-case complexity. The notion of generic-case complexity put on firm theoretical footing the results of various computer experiments with group-theoretic decision problems. Using results about random walks, the paper [14] provided formal proofs showing that often, under rather mild assumptions, various classic algorithmic decision problems in group theory, such as the word problem, the conjugacy problem and the subgroup membership problem have low (e.g., linear, quadratic, polynomial) generic-case complexity, even where the worst-case complexity is high or undecidable. The later work of Schupp and Kapovich [17], [20], [21] obtained similar results for the isomorphism problem (for one-relator groups and the quotients of the modular group), even though the isomorphism problem has been traditionally thought of as particularly intractable. In recent years, the study of generic-case complexity has become an active direction in group theory, and it also expanded to other contexts, such as the semi-group theory and general computability theory (e.g., see, [11], [30], [34]).

The theme of genericity in Paul's work naturally developed in the more algebraic direction as well. The study of properties of "generic" (in various senses) finitely presented groups was initiated by Gromov and Ol'shanskii in 1990s [4], [9], [31]. In particular, word-hyperbolicity turned out to be a ubiquitous phenomenon. Much of Schupp's "genericity" work concerned studying finer and more delicate algebraic properties for various group-theoretic objects (groups, individual group elements, subgroups, Nielsen-equivalence classes, etc.). In particular, this work led to the discovery of new and fascinating algebraic phenomena of Mostow-type isomorphism rigidity for generic one-relator groups [21]. This, in turn, allowed Kapovich and Schupp to obtain sharp asymptotic results [18], [21] about counting isomorphism types of one-relator groups on a fixed number of generators, where the length of the defining relation goes to infinity. Hitherto no results of this type were known in the context of groups given by finite presentations, in large part because of the difficulty of the isomorphism problem. Further work of Schupp, with various coauthors, explored a variety of group-theoretc genericity questions, such as generic stretching factors of free group automorphisms and generic subgroup distortion [13]; connections with Kolmogorov complexity [18]; number-theoretic properties of random elements in free groups [16], and so on. This direction in Paul's research remains very much a work in progress and he is actively pursuing a number of new projects there.

Paul Schupp's research, from his early work in 1960s to his latest results in the last few years, helped shape the birth and development of Geometric Group Theory and, deservedly, his name is among the most recognizable ones in the subject.

3. A few personal remarks

When I came to the University of Illinois as a newly hired Assistant Professor in the Fall of 2000, I already had great respect for Paul Schupp and his mathematical work. He was a classic and rather an iconic figure in the subject of Geometric Group Theory; back as an undergraduate student in Novosibirsk, I started learning about the subject by reading (around 1990–1991) the Lyndon and Schupp book, and it significantly influenced my mathematical education. So, understandably, I was initially somewhat intimidated. However, I quickly discovered that Paul was an absolute delight to communicate with and talk to about mathematics, as well as, as Douglas Adams would put it, about “life, universe and everything”.

He greatly impressed me by his intellectual curiosity and imagination, an abiding passion for mathematical research and extremely broad mathematical horizons. A true mathematical “Renaissance man”, Paul has active and genuine interest in a wide variety of topics in mathematics and adjacent subjects (complexity theory, logic, mathematical biology, physics and applied mathematics, just to name a few). I don't know anyone else who went to so many different seminars, both in our and other departments. I would often see Paul in the library enthusiastically perusing a newly arrived book on some far away topic, such as wavelets, cosmology, or stochastic processes (into which I probably would not have dared to look myself) and then excitedly tell me about it. It was also a great pleasure to be able to talk, in depth and detail, to a colleague every day about mathematics directly in my area of research, particularly to a colleague as enthusiastic and optimistic as Paul.

We quickly developed a mathematical rapport, which led to a highly successful and rewarding collaboration between us. It turned out that our mathematical interests and knowledge nicely complemented each other, while also having a large enough overlap to serve as a basis for joint work. I learned a great deal from Paul about many topics where my own background was limited, particularly complexity and computability theory. His passion about computer experiments as an important component of mathematical research, also significantly influenced my thinking. Most importantly, his optimism and enthusiasm about various potential mathematical approaches and ideas that we discussed was instrumental in getting quite a few of our joint projects moving. It has really been an honor, a pleasure and a privilege to work with Paul for the last ten years. I am very happy to see that even after his retirement, Paul continues thinking about mathematics and working on new projects.

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Ilya Kapovich, Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: kapovich@math.uiuc.edu