A NOTE ON THE POÉNARU CONDITION

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Abstract. We observe that a finitely generated group satisfying the Poénaru condition $P(2)$ is finitely presentable and has word problem solvable in exponential time.

1. Introduction

The following notion was suggested by V.Poénaru in [15] and further developed by V.Poénaru in [16] as well as by V.Poénaru and C.Tanasi in [17].

Definition 1. Let G be a finitely generated group and let X be a finite generating set of G. Let $\Gamma(G,X)$ be the Cayley graph of G with respect to X and let d_X denote the word-metric on $\Gamma(G,X)$.

For an integer $n \geq 1$ the group G is said to satisfy the *Poénaru condition $P(n)$ with respect to the generating set X* provided the following holds.

There exists a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that

(1) For any number $A > 0$ we have
$$\lim_{k\to\infty} k - Af(k) = \infty.$$

(2) Suppose $a, b \in G$ are elements at distance k from the identity element of G in the Cayley graph $\Gamma(G,X)$ such that $d_X(a,b) \leq n$. Then there is a path α of length at most $f(k)$ from a to b in the Cayley graph $\Gamma(G,X)$ such that α is contained in the closed ball of radius k around 1 in $\Gamma(G,X)$.

In fact V.Poénaru formulated this condition only for the functions of the type $f(k) = Ak^t + B$ where $0 \leq t < 1$. The above definition was first explicitly stated by L.Funar in [5]. It is worth noting that in any finitely generated group any two elements a and b at distance k from the identity can be joined inside the ball of radius k by a path of length $2k$. This explains why it is natural to require the function f in the Poénaru condition to be sub-linear.

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The Poénaru condition is a more general form of the *almost convexity condition* $AC(n)$ introduced by J.Cannon in [2]. One obtains condition $AC(n)$ by putting the function $f(k) = const$ in the definition above. It is not hard to see that $AC(2)$ implies $AC(n)$ for every $n \geq 2$. For this reason a group satisfying $AC(2)$ is called *almost convex* (see [2], [4], [18], [10] for more information). A serious drawback of almost convexity is that a group G may be almost convex with respect to one generating set but not another [20]. At present no such examples are known for $P(n)$, although they probably do exist.

Condition $P(n)$ turned out to be a useful tool for proving that various groups are simply connected at infinity. In fact, the original papers of V.Poénaru [15], [16] relate condition $P(n)$ to the long-standing conjecture that every closed 3-manifold with infinite fundamental group has universal covering homeomorphic to $\mathbb{R}^3$ (see also [12], [13]). Condition $P(n)$ is also related to Casson's condition $C(\alpha)$ [9], [17], [16].

It is easy to show and has been known for a long time that almost convex groups are not just finitely generated but finitely presentable. The main result of the present paper states that $P(2)$-groups are also finitely presentable. In fact, it turns out that a considerably weaker condition than $P(2)$ already implies finite presentability.

Definition 2. Let G be a group generated by a finite set X. Let $n \geq 2$ be an integer. We will say that G satisfies *condition* $K(n)$ if there is $k_0 \geq 1$ with the following property. Suppose $k \geq k_0$ and $a, b \in G$ are elements at distance k from the identity element of G in the Cayley graph $\Gamma(G, X)$ such that $d_X(a, b) \leq n$. Then there is a path α of length less that $2k$ from a to b in $\Gamma(G, X)$ such that α is contained in the closed ball of radius k around 1 in $\Gamma(G, X)$. Similarly, we define *condition* $K'(n)$ by requiring the length of α be less than $2k - 1$.

It is clear that $K'(n)$ implies $K(n)$ and that $P(n)$ implies $K'(n)$ and $K(n)$. As we observed before, any elements a and b at distance k from 1 can always be connected by a path of length $2k$ inside the ball of radius k around 1. Thus if condition $K(n)$ is relaxed any further, we would get the class of all finitely generated groups. Therefore we think of $K(n)$ as the "minimally restrictive" almost convexity condition.

Theorem 3. *Let G be a finitely generated group which satisfies condition $K(2)$ with respect to some finite generating set X. Then G is finitely presentable.*

This of course immediately implies that a $P(2)$-group is always finitely presentable.

Like the condition $AC(2)$, the Poénaru condition $P(2)$ and even the much weaker condition $K'(2)$ allows one to construct balls of arbitrarily large radius in the Cayley graph of G. Thus a $P(2)$ or even a $K'(2)$ group has solvable word problem. The proof is rather straightforward and follows the proof of J.Cannon in the almost convex case [2]. A careful analysis of the argument implies the following:

Theorem 4. *Let G be a finitely generated group which satisfies condition $K'(2)$ with respect to some finite generating set X.*
Then:

1. *The group G is finitely presentable and has solvable word problem.*
2. *The computational complexity of the word problem in G is at most exponential. That is there is $C > 1$ and an algorithm solving the word-problem in G with time-complexity at most C^n (where n is the length of the word in X being tested).*

By Theorem 4 any $P(2)$-group has solvable word problem and hence it also has a recursive isoperimetric function. Moreover exponential estimate is a fairly tight restriction for the complexity of the word problem. There are many examples (see for instance [1]) of finitely presentable groups with solvable word problem where the complexity of the word problem is strictly higher than exponential (indeed it can be made higher than any given reasonable recursive function). By Theorem 4 any such group is not $K'(2)$ and therefore not $P(2)$. Moreover, Theorem 4 implies that finitely generated groups with higher than exponential complexity of the word problem cannot embed in $K'(2)$ (and hence in $P(2)$) groups.

In [10], [11] S.Hermiller and J.Meier obtained some results related to isoperimetric functions of almost convex and, more generally, "tame combable" groups in the sense of [14]. S.Hermiller and J.Meier also showed in [11] that finitely presentable $P(3)$-groups are tame combable under some extra assumptions. It would be interesting to push these results by investigating further isoperimetric functions of groups satisfying $P(2)$ and relating them to the function $f(k)$ from Definition 1.

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2. Proof of the main result.

In this section we will establish Theorem 3.

Let G be a group generated by a finite set X such that G satisfies condition $K(2)$ with respect to X. If τ is a path or a word, we will denote by $|\tau|$ the length of τ. We will also denote by d_X the word-metric on the Cayley graph $\Gamma(G,X)$. Note that the label of every edge-path in $\Gamma(G,X)$ is a word in $X \cup X^{-1}$. If the edge-path is reduced (that is has no backtracks), its label is a freely reduced word.

Let k_0 be the constant from the definition of Condition $K(2)$. Put $k_1 = k_0 + 1$. Let

$$R = \{w \in F(X) \,|\, w =_G 1, |w| \le 2k_1 + 1\}.$$

Clearly R is finite since X is finite. We will show that in fact G has the finite presentation $G = \langle X|R\rangle$. To see this it suffices to to establish:

Claim. Suppose $w \in F(X)$ is a freely reduced word in X which represents 1 in G. Then the relation $w =_G 1$ follows from R, that is to say w belongs to the normal closure of R in $F(X)$.

We will prove the above Claim by induction on $|w|$. When $|w| \le 2k_1 + 1$, this is obvious since in this case $w \in R$. Suppose now that $|w| > 2k_1 + 1$ and that the Claim has been verified for all shorter relations in G. We may assume that no letter of w represents 1 in G. Indeed, suppose $w = w'sw''$ where $s \in X \cup X^{-1}$ and $s =_G 1$. Then the relation $w'w'' =_G 1$ can be derived from R by the inductive hypothesis since $|w'w''| < |w|$. Moreover $s \in R$ and $w = w'sw'' = w's(w')^{-1}(w'w'')$ and hence $w =_G 1$ can be derived from R as well. Therefore from now on we will assume that every letter of w represents a nontrivial element of G. We can also assume that w is not just freely reduced but also cyclically reduced. We will need to treat the cases when $|w|$ is even and odd separately.

Case 1. Suppose $|w|$ is even. Then $|w| = 2k$ where $k > k_1$, since $|w| > 2k_1 + 1$. Hence $k - 1 > k_1 - 1 = k_0$.

Let σ be the path in $\Gamma(G,X)$ starting at 1 and with label w. Since $w =_G 1$, the path σ is in fact a loop based at $1 \in G$. For every vertex v on σ let ρ_v be a geodesic path from 1 to v. Let $[v]$ be the label of ρ_v, so that $[v]$ is an X-geodesic word representing $v \in G$. To see that $w =_G 1$ follows from R it suffices to show that for every edge e on σ from u to u' and labeled by $s \in X \cup X^{-1}$ the relation $r_e := [u]s[u']^{-1} =_G 1$ follows from R. Since $|w| = |\sigma| = 2k$, for every vertex v on σ we have $|\rho_v| \le k$. Moreover, if e is an edge on σ from u to u' and label $s \ne_G 1$ then one of the following holds:
(a) Both $|\rho_u| < k$ and $|\rho_{u'}| < k$.
(b) One of the paths $\rho_u, \rho_{u'}$ has length k and the other has length $k-1$.

If (a) occurs then the length of the relation $r_e = [u]s[u']^{-1}$ is at most $(k-1) + (k-1) + 1 = 2k - 1 < 2k = |w|$. Therefore the relation $r_e =_G 1$ follows from R by the inductive hypothesis.

Suppose now that (b) happens. Without loss of generality we may assume that $|\rho_u| = k-1$ and $|\rho_{u'}| = k$. Recall that $u' =_G us$ and $d_X(u,u') = 1$. Let v be the vertex on $\rho_{u'}$ at distance 1 from u'. Thus $d_X(v,u) \leq 2$. Since $\rho_{u'}$ is a geodesic, we have $d_X(1,v) = k-1$. Also by assumption $d_X(1,u) = k-1$. Since $k-1 \geq k_0$, by condition $K(2)$ there is an edge-path path α from u to v such that the length of α is at most $2(k-1)-1 = 2k-3$ and such that α is completely contained in the ball of radius $k-1$ around the identity. Clearly, we may assume that α is an edge-path without backtracks. Let $z \in F(X)$ be the label of the path α. Also let $[v]$ be the initial segment of the word $[u']$ of length $k-1$, so that $[v]$ is a geodesic representative of $v \in G$. Let s' be the last letter of $[u']$. To show that $r_e = [u]s[u']^{-1} =_G 1$ follows from R, it suffices to show that both $[u]z[v]^{-1} =_G 1$ and $zs'(s)^{-1}1 =_G 1$ follow from R.

We have

$$|zs'(s)^{-1}| = |z| + 2 \leq 2k-3+2 = 2k-1 < |w|$$

Thus the relation $zs'(s)^{-1}1 =_G 1$ follows from R by the inductive hypothesis.

It remains to show that $[u]z[v]^{-1} =_G 1$ follows from R. For each vertex v' on α different from u,v choose a geodesic word $[v']$ representing $v' \in G$. For any edge e' on α with endpoints v', v'' and label s'' we have $d_X(1,v') \leq k-1$ and $d_X(1,v'') \leq k-1$ since α is contained in the ball of radius $k-1$ around 1 . Therefore $|[v']s''[v'']^{-1}| \leq 2(k-1)+1 = 2k-1 < 2k = |w|$. Hence the relation $[v']s''[v'']^{-1} =_G 1$ follows from R by the inductive hypothesis. Since this was true for each edge e' on α, we conclude that the relation $[u]z[v]^{-1} =_G 1$ also follows from R. This concludes the analysis of Case 1.

Case 2. Suppose now that $|w|$ is odd. Then $|w| = 2k+1$ where $k > k_1$ since $|w| > 2k_1+1$. Thus $k > k_1 > k_0$.

We define the path σ as in Case 1. Also, for each vertex v on σ we define ρ_v and $[v]$ as in Case 1.

The fact that $|w| = 2k+1$ implies $|\rho_v| \leq k$ for each vertex v on σ. Once again, it suffices to show that for each edge e on σ from u to u' and with label s the relation $r_e = [u]s[u']^{-1} =_G 1$ follows from R. Since $|w| = 2k+1$, for every e as above one of the following holds:

(a) Both ρ_u and $\rho_{u'}$ have length at most $k-1$.

(b) One of ρ_u, $\rho_{u'}$ has length k and the other has length $k-1$.

(c) Both ρ_u, $\rho_{u'}$ have length k.

In case (a) the relation r_e has length at most $2(k-1)+1 = 2k-1 < 2k+1 = |w|$. In case (b) the relation r_e has length $k+(k-1)+1 =$

$2k < 2k+1 = |w|$. Thus in both of these cases $r_e =_G 1$ follows from R by the inductive hypothesis.

Suppose now that (c) occurs. Thus $u' =_G us$, $d_X(u,u') = 1$ and $d_X(1,u) = d_X(1,u') = k$. Since $k > k_0$, by condition $K(2)$ there is a reduced edge-path α from u to u' of length at most $2k-1$ such that α is contained in the ball of radius k around 1. Let $z \in F(X)$ be the label of α. It suffices to show that the relations $[u]z[u']^{-1} =_G 1$ and $zs^{-1} =_G 1$ follow from R. We have

$$|zs^{-1}| = |\alpha| + 1 \leq 2k - 1 + 1 = 2k < 2k + 1 = |w|.$$

Thus $zs^{-1} =_G 1$ follows from R by the inductive hypothesis.

It remains to deal with the relation $[u]z[u']^{-1} =_G 1$. Recall that $d_X(1,u) = d_X(1,u') = k$ and that α is contained in the ball of radius k around the identity. For each vertex v on α different from u,u' choose a geodesic word $[v]$ representing $v \in G$.

Let e' be an arbitrary edge on α from v' to v'' and with label s. Again, it is enough to show that the relation $[v']s[v'']^{-1} =_G 1$ follows from R. Note that the edge e' and the vertices v', v'' are contained in the ball of radius k around 1. Hence $d_X(1,v') \leq k$ and $d_X(1,v'') \leq k$. We claim that at least one of these inequalities is strict. Indeed, suppose that $d_X(1,v') = d_X(1,v'') = k$. Let p be the midpoint of the edge e' joining v' to v''. Then by the definition of the distance function for $\Gamma(G,X)$ we have $d(1,p) = k + \frac{1}{2}$. However, $k + \frac{1}{2} > k$, which contradicts the fact that e' is contained in the ball of radius k around 1.

Thus either $d_X(1,v') \leq k-1$ or $d_X(1,v'') \leq k-1$. Therefore

$$|[v']s[v'']^{-1}| \leq (k-1) + k + 1 = 2k < 2k + 1 = |w|$$

and hence the relation $[v']s[v'']^{-1} =_G 1$ follows from R by the inductive hypothesis.

This completes the proof of the Claim and thus of Theorem 3.

3. The Poénaru Condition and the Word Problem

In this section we will establish Theorem 4. Let G be a group generated by a finite set X such that G satisfies condition $K'(2)$ with respect to X. We will describe an algorithm which constructs a sequence of balls $B(0), B(1), B(2), \dots, B(n), \dots$ of radius n around the identity in $\Gamma(G,X)$. This will clearly imply that the word problem in G is solvable.

Let k_0 be the constant from the definition of $K'(2)$. We first take the ball $B(k_0)$ in $\Gamma(G,X)$. [This may be done since any fixed finite amount of data can be used as a part of an algorithm.]

Suppose now that $k \geq k_0$ and that the balls $B(n), 0 \leq n \leq k$ have already been constructed. We need to build $B(k+1)$. Let $S(k)$ be the sphere of radius k, that is the set of elements of G at distance k from 1 in $\Gamma(G, X)$. For each $x \in X \cup X^{-1}$ and each $a \in S(k)$ we add a directed edge labeled x with origin a to $B(k)$ provided there is no such edge in $B(k)$ already. Every new vertex represents an element at distance k or $k+1$ from 1. Thus we have to decide which of the new vertices need to be identified with vertices from $S(k)$ or with other new vertices. Suppose v and v' are two new vertices which represent the same element $g \in G$ with $d_X(1, g) = k+1$. Let v be the endpoint of a new edge originating from $a \in S(k)$ and with label x. Let v' be the endpoint of a new edge originating from $a' \in S(k)$ and with label x'. Then $d_X(a, a') \leq 2$. Therefore by condition $K'(2)$ there is a reduced edge-path α from a to a' in $B(k)$ such that $|\alpha| \leq 2k-2$. Let z be the label of α. Then the word $zx'(x)^{-1}$ is a relation in G of length at most $2k$. Therefore $zx'(x)^{-1}$ can be read as a label of a loop based at 1 in $B(k)$. Therefore we can decide which of the new vertices need to be identified with each other.

Indeed, first we make a list L_k of all the words of length at most $2k-2$ in G. For each pair of new edges $e = (a, v)$ and $e' = (a', v')$ with labels x and x' we then inspect the words from L_k one by one and check if there is a path from a to a' in $B(k)$ with a label from L_k. If not, the vertices v and v' need not be identified. If yes, we find a specific word $z \in L_k$ of length at most $2k-2$ such that z is the label of a path in $B(k)$ from a to a'. Then the word $w = zx'(x)^{-1}$ has length at most $2k$. We check whether or not w is the label of a loop based at 1 in $B(k)$. If not, we leave the vertices v and v' distinct. If yes, we identify v and v' into a single vertex.

A similar process allows us to check which of the new vertices need to be identified with vertices of $S(k)$. After this task is complete, we have constructed the ball $B(k+1)$ of radius $k+1$ in $\Gamma(G, X)$, as required. Thus we have established that the word problem in G is solvable.

We will now establish part (2) of Theorem 4. Clearly it suffices to show that for some constant $A > 1$ each transition from the ball $B(k)$ to $B(k+1)$ requires at most A^k steps. Indeed, if this is true, then $B(k)$ can be constructed in at most

$$1 + A + A^2 + \cdots + A^{k-1} = \frac{A^k - 1}{A - 1} \leq \frac{A^k}{A - 1}$$

which implies statement (2) of Theorem 4 regarding the computational complexity of the word problem. Thus we need to establish the existence of A.

Let s be the number of elements in $X \cup X^{-1}$, so that $s \geq 2$. For the transition from $B(k)$ to $B(k+1)$ (where $k \geq k_0$) we first need to make the list L_k of all the words in $X \cup X^{-1}$ of length at most $2k-2$. There are at most s^{2k} such words. The number of the vertices in the sphere $S(k)$ is at most s^k. Each vertex on $S(k)$ has at most s new edges. Thus the number of new vertices is at most s^{k+1}. We need to inspect each pair v, v' of new vertices for possible identification. Similarly we need to inspect each pair v, a where v is a new vertex and a is a vertex on $S(k)$. The total number of pairs to be inspected for possible identification is at most

$$\frac{s^{k+1}(s^{k+1}-1)}{2} + \frac{s^{k+1}(s^{k+1}-1)}{2} \leq s^{2k+2}.$$

For each pair of vertices being tested for possible identification we need to check if some word z from L_k is the label of a path between two specific vertices on $S(k)$. This requires checking at most s^{2k} words. Finally, once such z is found, we must check if a specific word of length at most $2k$ is the label of a loop based at 1 in $B(k)$. This can be done in no more than $2k$ steps. Thus the transition from $B(k)$ to $B(k+1)$ requires at most

$$s^{2k+2}(s^{2k}+2k) \leq s^{2k+2}s^{2k+1} = s^{4k+3}$$

steps, provided k is such that $2k \leq s^{2k}$. Since $s \geq 2$, this last inequality holds for all sufficiently big k. Thus part (2) of Theorem 4 is proved.

References

[1] J.-C. Birget, A.Yu. Ol'shanskii, E.Rips and M.Sapir, *Isoperimetric functions of groups and computational complexity of the word problem,* preprint, 1998

[2] J. W. Cannon, *Almost convex groups,* Geom. Dedicata **22**(1987), no. 2, 197–210

[3] D. E. Cohen, *Isodiametric and isoperimetric inequalities for group presentations,* Internat. J. Algebra Comput. **1** (1991), no. 3, 315–320.

[4] J. W. Cannon, W. J. Floyd, M. A. Grayson, and W. P. Thurston, *Solvgroups are not almost convex,* Geom. Dedicata **31** (1989), no. 3, 291–300

[5] L. Funar, *On discrete solvgroups and Poénaru's condition,* Arch. Math. (Basel) **72** (1999), no. 2, 81–85

[6] S. M. Gersten, *The double exponential theorem for isodiametric and isoperimetric functions,* Internat. J. Algebra Comput. **1** (1991), no. 3, 321–327

[7] S. M. Gersten, *Dehn functions and l_1-norms of finite presentations,* in "Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)", 195–224, Math. Sci. Res. Inst. Publ., **23**, Springer, New York, 1992

[8] S. M. Gersten, *Isoperimetric and isodiametric functions of finite presentations,* in "Geometric group theory, Vol. 1 (Sussex, 1991)", 79–96, London Math. Soc. Lecture Note Ser., **181** , Cambridge Univ. Press, Cambridge, 1993

[9] S. M. Gersten and J. R. Stallings, *Casson's idea about* 3*-manifolds whose universal cover is* R^3*,* Internat. J. Algebra Comput. **1** (1991), no. 4, 395–406

[10] S. Hermiller and J. Meier, *Tame combings, almost convexity and rewriting systems for groups,* Math. Z. **225** (1997), no. 2, 263–276

[11] S. Hermiller and J. Meier, *Measuring the tameness of almost convex groups,* Trans. Amer. Math. Soc. **353** (2000), no. 3, 943–962

[12] M. Mihalik, *Compactifying coverings of* 3*-manifolds,* Comment. Math. Helv. **71** (1996), no. 3, 362–372

[13] M. Mihalik, *Semistability at infinity, simple connectivity at infinity and normal subgroups,* Topology Appl. **72** (1996), no. 3, 273–281

[14] M. Mihalik and S. Tschantz, *Tame combings of groups,* Trans. Amer. Math. Soc. **349** (1997), no. 10, 4251–4264

[15] V. Poénaru, *Almost convex groups, Lipschitz combing, and* π_1^∞ *for universal covering spaces of closed* 3*-manifolds,* J. Differential Geom. **35** (1992), no. 1, 103–130

[16] V. Poénaru, *Geometry " la Gromov" for the fundamental group of a closed* 3*-manifold* M^3 *and the simple connectivity at infinity of* $\tilde{M}^3$*,* Topology **33** (1994), no. 1, 181–196

[17] V. Poénaru and C. Tanasi, *Hausdorff combing of groups and* π_1^∞ *for universal covering spaces of closed* 3*-manifolds,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), no. 3, 387–414

[18] M. Shapiro and M. Stein, *Almost convex groups and the eight geometries,* Geom. Dedicata **55** (1995), no. 2, 125–140

[19] J. R. Stallings and A. R. Wolf, *The Todd-Coxeter process, using graphs,* in "Combinatorial group theory and topology (Alta, Utah, 1984)", 157–161, Ann. of Math. Stud., **111**, Princeton Univ. Press, Princeton, NJ, 1987

[20] C. Thiel, *Zur fast-Konvexität einiger nilpotenter Gruppen. (German) [On the almost convexity of some nilpotent groups],* Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, (1991); Bonn Mathematical Publications, vol. 234. Universität Bonn, Mathematisches Institut, Bonn, 1992

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