MAPPING TORI OF ENDOMORPHISMS OF FREE GROUPS

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Abstract. For a large class of endomorphisms of finitely generated free groups we prove that their mapping tori groups are word-hyperbolic if and only if they don't contain Baumslag-Solitar subgroups.

1. Introduction

In this paper we study mapping tori of injective endomorphisms of free groups. For a free group $F(X)$ and an injective endomorphism $\phi : F(X) \longrightarrow F(X)$ the following group is called the *mapping torus* of ϕ and is denoted M_ϕ:

$$M_\phi = \langle t, X \,|\, t^{-1}xt = \phi(x) \text{ for every } x \in X\rangle \tag{1}$$

Note that M_ϕ is an "ascending" HNN-extension of $F(X)$ with associated subgroups $F(X)$ and $\phi(F(X))$.

When ϕ is an automorphism of $F(X)$, the subgroup $F(X)$ is normal in M_ϕ and there is a short exact sequence

$$1 \longrightarrow F(X) \longrightarrow M_\phi \longrightarrow \mathbb{Z} \longrightarrow 1$$

Mapping tori of automorphisms of finitely generated free groups have been extensively studied and by now are sufficiently well understood. Their structure closely resembles that of mapping tori of automorphisms of fundamental groups of closed surfaces.

Thus mapping tori of automorphisms of free groups are very similar to fundamental groups of closed 3-manifolds fibering over a circle. Moreover, the study of mapping tori of automorphisms of free groups requires developing an analogue of Thurston's classification of homeomorphisms of closed surfaces. Results of this type for $Out(F_n)$ may be found in [BH92],[CV86], [Lus92], [Sel96] and other papers.

However, very little is known about the structure of mapping tori of arbitrary injective endomorphisms of free groups which are not necessarily "onto". And yet, this class of groups also appears to be very interesting and worth-while to investigate. One one hand, it is a natural extensions of mapping tori of automorphisms (which, as we have seen, play an important role in 3-dimensional topology). On the other hand, this class also generalizes the so-called solvable Baumslag-Solitar groups. If in (1) X has just one element, say x, then $F(X)$ is infinite cyclic, $\phi(x) = x^n$, $n \neq 0$ and

$$M_\phi = \langle t, x \,|\, t^{-1}xt = x^n\rangle$$

The group with the above presentation is called the Baumslag-Solitar group $B(1, n)$ (it is easily seen to be metabelian, that is solvable of step two). Algebraic and geometric properties of metabelian Baumslag-Solitar groups also have been the subject of extensive studies in Group Theory. One of the recent fascinating results in this area is the work of B.Farb and L.Mosher [FM98] on classifying quasi-isometry types of these groups.

Among the few known facts about mapping tori of arbitrary injective endomorphisms of finitely generated free groups is a recent remarkable theorem of M.Feighn and M.Handel [FH97]. This result states that such groups are *coherent*, that is all their finitely generated subgroups are finitely presentable. Some interesting structural results about endomorphisms of free groups were obtained by W.Dicks and E.Ventura [DV96], R.Goldstein and E.Turner [GT86] and other authors.

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In the present paper we address the question of hyperbolicity of groups given by presentation (1). If ϕ is an automorphism of $F(X)$, then it is completely understood when M_ϕ is word-hyperbolic. The train-track theory of M.Bestvina and M.Handel [BH92] together with the Combination Theorem for word-hyperbolic groups by M.Bestvina and M.Feighn [BF92] imply the following.

Theorem 1.1. [BF92] *Let ϕ be an automorphism of a finitely generated free group $F(X)$. Then the following conditions are equivalent.*

1. *The group M_ϕ is word-hyperbolic.*
2. *The group M_ϕ does not contain subgroups isomorphic to $\mathbb{Z}\times\mathbb{Z}$.*
3. *The automorphism ϕ has no periodic conjugacy classes (that is there is no $f\in F(X), f\neq 1$, $k\neq 0$ such that $\phi^k(f)$ is conjugate to f in $F(X)$).*

A natural conjecture for mapping tori of arbitrary injective endomorphisms is that M_ϕ is word-hyperbolic if and only if M_ϕ has no subgroups isomorphic to $B(1,n)$ if and only if ϕ has no periodic conjugacy classes (that is there is no $f\in F(X), f\neq 1$, $k>0, n\neq 0$ such that $\phi^k(f)$ is conjugate to f^n in $F(X)$). This conjecture appears to be very difficult. On one hand, it should include the automorphism case and Theorem 1.1. On the other hand, $F(X)$ has many more endomorphisms than automorphisms. The general structure and dynamics of endomorphisms appears to be much more complicated and substantially different from that of automorphisms.

In the present paper we prove the above conjecture for a sufficiently large and interesting class of endomorphisms.

Namely, we say that an endomorphism $\phi : F(X)\longrightarrow F(X)$ is an *immersion* if for every $x,y\in X\cup X^{-1}$ such that $y\neq x^{-1}$ the word $\phi(x)\phi(y)$ is freely reduced as written. For instance if for every $x\in X$ the word $\phi(x)$ begins with the letter x and ends with the letter x then ϕ is an immersion (we will call such ϕ a *simple immersion*) . Our main result is the following statement.

Theorem A (c.f. Theorem 5.5)**.** *Let ϕ be an immersion of a finitely generated free group $F(X)$. Then the following conditions are equivalent.*

1. *The group M_ϕ is word-hyperbolic.*
2. *The group M_ϕ does not contain subgroups isomorphic to $B(1,p)$, $p>0$.*
3. *The automorphism ϕ has no periodic conjugacy classes, that is there are no $f\in F(X), f\neq 1$, $p>0$, $k>0$ such that $\phi^k(f)$ is conjugate to f^p in the group $F(X)$.*

It should be noted that Theorem A is connected to proving the so-called BS-conjecture for one-relator groups. This difficult (and still open) conjecture states that a finitely generated one-relator group is word-hyperbolic if and only if it does not contain Baumslag-Solitar subgroups $B(m,n)=\langle a,b\,|\,b^{-1}a^nb=a^m\rangle$, $m,n\neq 0$. As it was observed by the author in [K98], one of the most difficult cases in proving this conjecture is dealing with one-relator groups which arise as mapping tori groups of endomorphisms of free groups.

One of the immediate corollaries of Theorem A is:

Corollary B (c.f. Corollary 5.6)**.** *There is an algorithm which, for an immersion ϕ of a finitely generated free group $F(X)$ decides whether or not the mapping class group M_ϕ is word-hyperbolic.*

We also observe that there is a strict dichotomy for isoperimetric functions of mapping tori of simple immersions.

Corollary C (c.f. Corollary 5.7)**.** *Let ϕ be a simple immersion of a finitely generated free group $F(X)$ such that for every $x\in X$ $\phi(x)\neq x$. Then the isoperimetric function of M_ϕ is either linear or strictly exponential.*

One of the main technical tools is Proposition 3.7 which analyzes forward images $\phi^n(F(X))$ $n\geq 1$ of a simple immersion ϕ and shows that they are, in a certain sense, close to being malnormal in $F(X)$. This allows us to use the Combination Theorem of M.Bestvina and M.Feighn to deduce Theorem A.

In the last section we list some of interesting open problems for mapping tori of endomorphisms.

2. Basic properties of mapping tori.

If G is a group and Y is a subset of G, we will denote by $\langle Y \rangle$ the subgroup of G generated by Y.

For the remainder of this section let $F(X)$ be a free group of finite rank on X and let $\phi : F(X) \longrightarrow F(X)$ be an injective endomorphism of $F(X)$.

Definition 2.1 (Periodic conjugacy class)**.** Let $\phi : G \longrightarrow G$ be an endomorphism of a group G. We say that $g \in G$, $g \neq 1$ represents a *periodic conjugacy class* of ϕ if there exist $k, p \in \mathbf{Z}$, $f \in G$ such that $g^p \neq 1, k, p \geq 1$ and

$$f^{-1}\phi^k(g)f = g^p$$

in G.

Lemma 2.2. *Let $n \geq 1$ and let $H = \langle t^n, X \rangle \leq M_\phi$. Then the following holds.*

1. *For $s = t^n$ the subgroup H is of index n in M_ϕ and H has the following presentation on s, X:*

$$\langle s, X \mid s^{-1}xs = \phi^n(x) \text{ for every } x \in X \rangle$$

In particular, $H \cong M_{\phi^n}$.

2. *Let $h \in F(X)$ and let $\psi : F(X) \longrightarrow F(X)$ be an endomorphism of $F(X)$ defined as $\psi(x) = h^{-1}\phi(x)h$ for every $x \in X$. Let $s = th$. Then $M_\phi = \langle t, X \rangle = \langle s, X \rangle$ and M_ϕ has the presentation on s, X:*

$$\langle s, X \mid s^{-1}xs = \psi(x) \text{ for every } x \in X \rangle$$

In particular, $M_\phi \cong M_\psi$.

Proof. The proof is an elementary exercise and is left to the reader. □

Lemma 2.3. *Suppose there exist $g \in F(X), g \neq 1$, $m, n \neq 0$, $l \geq 1$ such that $\phi^l(g^n)$ is conjugate to g^m in $F(X)$.*

Then there exist $f \in F(X)$, $f \neq 1$, $k \geq 1$, $p > 0$, $h \in F(X)$ such that

1. *$\phi^k(f) = h^{-1}f^p h$ in $F(X)$ (and so ϕ has a non-trivial conjugacy class).*
2. *The subgroup $H = \langle t^k h^{-1}, f \rangle \leq M_\phi$ has the presentation on $s = t^k h^{-1}, f$:*

$$\langle s, f | s^{-1}fs = s^p \rangle, \tag{2}$$

that is $H \cong B(1, p)$.

Proof. Suppose there exist $f, h \in F(X)$, $f \neq 1$, $k > 0$, $m \neq 0, n \neq 0$ such that

$$\phi^k(f^n) = h^{-1}f^m h \tag{3}$$

First note that $\phi^{2k}(f^{n^2}) = \phi^k(h)^{-1}h^{-1}f^{m^2}h\phi^k(h)$. Hence we may assume that $n > 0, m > 0$ in (3).

Put $\psi = \phi^k$.

Thus there exist $f, h \in F(X)$, $f \neq 1$, $m > 0, n > 0$ such that

$$\psi(f^n) = h^{-1}f^m h \tag{4}$$

Put $\theta(g) = h\psi(g)h^{-1}$ for every $g \in F(X)$. Put $s = t^k h^{-1}$. By Lemma 2.2 the subgroup $L = \langle s, F(X) \rangle = \langle t^k, F(X) \rangle \leq M_\phi$ is of index k in M_ϕ and has the presentation on s, X:

$$\langle s, X \mid s^{-1}xs = \theta(x), x \in X \rangle$$

that is $L \cong M_\theta$. We will identify L and M_θ from now on.

Thus for some $f \in F(X)$, $f \neq 1$, $n > 0, m > 0$:

$$\theta(f^n) = f^m \tag{5}$$

We will take $f \neq 1$ to be an element of minimum length such that for some $n > 0, m > 0$ equation (5) holds. Thus f is not a proper power in $F(X)$ and generates a maximal cyclic subgroup of $F(X)$. We have

$$[\theta(f), \theta(f^n)] = 1, [\theta(f^n), f^m] = 1, [f^m, f] = 1.$$

Commutativity is a transitive relation on the set of nontrivial elements in a free group. Therefore $[\theta(f), f] = 1$. Since f generates a maximal cyclic subgroup of $F(X)$, this means that

$$\theta(f) = f^p \tag{6}$$

for some $p \neq 0$. Hence $\theta(f^n) = f^{pn} = f^m$, $m = pn$ and therefore $p > 0$.

Case 1. Suppose that $f \in \theta^i(F)$ for every $i \geq 1$. Let $K = \langle s, f \rangle \leq M_\theta$.

Claim 1. We claim that K has the presentation on f, s

$$\langle f, s \,|\, s^{-1} f s = f^p \rangle \tag{7}$$

Since $\theta(f) = f^p$, we have $s^{-1} f s = \theta(f) = f^p$ in M_θ. Thus the defining relation of (7) is satisfied by $f, s \in M_\theta$. The group defined by presentation (7) is an HNN-extension of the infinite cyclic group on f. To see that Claim 1 holds, it is enough to show that any normal form word with respect to the HNN-presentation (7) defines a nontrivial element of M_θ.

It is clear that any f, s-word which has nonzero exponent sum on s defines a nontrivial element of M_θ. Any nontrivial normal form w with respect to the HNN-presentation (7), which has zero s-exponent sum, can be re-written using the relation of (7) as

$$w = s^n f^q s^{-n}$$

where $n \geq 0$, $q \neq 0$. By our assumption $f \in \theta^n(F)$ and therefore $f = \theta^n(b)$ for some $b \neq 1$, $b \in F(X)$. Hence in M_θ we have

$$\begin{aligned} s^n f^q s^{-n} =& s^n \theta^n(b^q) s^{-n} = \\ =& b^q \neq_{M_\theta} 1 \text{ since } q \neq 0. \end{aligned}$$

Thus Claim 1 is verified and $K \cong B(1, p)$ as required.

Case 2. Suppose that $f \in \theta^N(F), f \notin \theta^{N+1}(F)$ for some $N \geq 0$. Then $f = \theta^N(a)$, $a \notin \theta(F)$ and

$$\phi^{N+1}(a) = \theta(\theta^N(a)) = (\theta^N(a))^p = \theta^N(a^p)$$
$$\theta(a) = a^p, \text{ where } p > 0, a \notin \phi(F)$$

Note that since f is not a proper power, a is also not a proper power.

Claim 2. For every $i, 0 < i < p$ we have $a^i \notin \theta(F)$.

Indeed, suppose $a^i = \theta(b), 0 < i < p$. Then

$$[\theta(b), \theta(a)] = 1 \text{ and hence } [a, b] = 1$$

Since a is not a proper power and generates a maximal cyclic subgroup of F, this means that $b = a^j$ that is $\theta(a^j) = a^i$ where $0 < i < p$. However $\theta(a) = a^p$ implies $\theta(a^j) = a^{jp}$. Thus $i = jp$ where $0 < i < p$ and j is an integer. This is obviously impossible. Thus Claim 2 holds.

In fact, Claim 2 implies that $a^i \in \theta(F)$ if and only if i is divisible by p. Then an elementary argument (similar to Case 1) shows that the subgroup $K = \langle s, a \rangle \leq M_\theta$ has the presentation on a, s

$$\langle a, s \,|\, s^{-1} a s = a^p \rangle$$

that is $K \cong B(1, p)$.

We have shown that in both Case 1 and Case 2 there exist $f \in F(X), f \neq 1, p > 0$ such that $\theta(f) = f^p$ and the subgroup $H = \langle s, f\rangle \leq M_\theta$ has the presentation on f, s

$$\langle f, s \mid s^{-1}fs = f^p\rangle$$

that is $K \cong B(1,p)$. Recall that $\psi = \phi^k$, $s = t^k h^{-1}$ and that $\theta(g) = h\psi(g)h^{-1} = h\phi^k(g)h^{-1}$ for every $g \in F(X)$. Also $L = \langle t^k, X\rangle = \langle t^k h^{-1} = s, X\rangle = M_\theta$ is a subgroup of index k in M_ϕ. Thus $\theta(f) = f^p$ implies $h\phi^k(f)h^{-1} = f^p$, $\phi^k(f) = h^{-1}f^p h$. Also we have verified that the subgroup H generated by $f, s = t^k h^{-1}$ is isomorphic to $B(1,p)$ on these generators.

Thus both conditions (1) and (2) of Lemma 2.3 hold.

□

3. The structure of simple immersions of free groups.

Let $F(X)$ be a free group on X. For an element $f \in F(X)$ we denote by $|f|_X$ (or just $|f|$) the length of the freely reduced word in X representing f. Similarly, we denote by $||f||_X$ (or just $||f||$) the X-length of the shortest element in the conjugacy class of f in $F(X)$. Note that $||f||$ is the length of the cyclically reduced form of f.

If two words w, v in X have the same freely reduced form we will write $w = v$. If w and v are equal as words, we will write $w \equiv v$.

Definition 3.1 (Immersion)**.** Let $F(X)$ be a free group on a finite basis X. Let $\phi : F(X) \longrightarrow F(X)$ be an endomorphism. We say that ϕ is an *immersion* with respect to X if for any $x, y \in X \cup X^{-1}$ such that $xy \neq 1$ we have

$$|\phi(xy)| = |\phi(x)| + |\phi(y)|$$

Notice that if ϕ is an immersion then for any freely reduced word $w = x_1 \dots x_n$, $x_i \in X \cup X^{-1}$ the word

$$\phi(x_1)\phi(x_2)\dots\phi(x_n)$$

is freely reduced as written and

$$|\phi(w)| = |\phi(x_1)| + \dots + |\phi(x_n)|.$$

Definition 3.2 (simple immersion)**.** Let $F(X)$ be a free group on a finite basis X. Let $\phi : F(X) \longrightarrow F(X)$ be an endomorphism. We say that ϕ is a *simple immersion* with respect to X if for each $x \in X$ the freely reduced form of $\phi(x)$ begins with x and ends with x.

It is easy to see that any simple immersion is an immersion. In fact, the converse is also "almost" true.

Lemma 3.3. *Let $F(X)$ be a free group on a finite basis X. Let n be the cardinality of X. Then for any immersion $\phi : F(X) \longrightarrow F(X)$ the endomorphism $\phi^{(2n)!}$ is a simple immersion.*

Proof. Let $\sigma_\phi : (X \cup X^{-1}) \longrightarrow (X \cup X^{-1})$ be defined as follows:
for each $x \in X \cup X^{-1}$ $\sigma_\phi(x)$ is the first letter of the freely reduced form of $\phi(x)$.

Since ϕ is an immersion, the map σ_ϕ is a permutation of $X \cup X^{-1}$. It is easy to see that for any $k \geq 1$ the endomorphism ϕ^k is an immersion and $\sigma_{\phi^k} = (\sigma_\phi)^k$. The cardinality of $X \cup X^{-1}$ is $2n$. Therefore the group of permutations of $X \cup X^{-1}$ has order $(2n)!$ and thus $(\sigma_\phi)^{(2n)!} = id$. This means, by the above remark, that $\sigma_{\phi^{(2n)!}} = id$.

Thus for each $x \in X \cup X^{-1}$ the word $\sigma_{\phi^{(2n)!}}(x)$ starts with x. Therefore for every $x \in X \cup X^{-1}$ the word $\sigma_{\phi^{(2n)!}}(x^{-1}) = [\sigma_{\phi^{(2n)!}}(x)]^{-1}$ starts with x^{-1}. This means that $\sigma_{\phi^{(2n)!}}(x)$ ends with x.

Thus the endomorphism $\sigma_{\phi^{(2n)!}}$ is a simple immersion.

□

The following proposition summarizes some basic properties of simple immersions.

Proposition 3.4. *Let $\phi : F(X) \longrightarrow F(X)$ be a simple immersion of a finitely generated free group $F(X)$. Then the following holds.*

1. *For any $f \in F(x), f \neq 1$ we have $|\phi(f)| \geq |f|$.*
2. *The map $\phi : F(X) \longrightarrow F(X)$ is injective.*
3. *The map $\phi : F(X) \longrightarrow F(X)$ is not onto unless $\phi = id_{F(X)}$.*
4. *For any $f \in F(X)$ the element $\phi(f)$ is cyclically reduced if and only if f is cyclically reduced.*
5. *For every $n \geq 1$ $\phi^n : F(X) \longrightarrow F(X)$ is also a simple immersion.*
   Assume in addition that $|\phi(x)| \geq 2$ for every $x \in X$. Then:
6. *For any $f \in F(x)$ we have $|\phi(f)| \geq 2|f|$.*
7. *For any $f \in F(X)$ we have $||\phi(f)|| \geq 2||f||$.*
8. *The map $\phi : F(X) \longrightarrow F(X)$ has trivial* stable image*, that is*

$$\cap_{n\geq 1}\phi^n(F(X)) = 1$$

Proof. The proof of this proposition is an elementary exercise and is left to the reader. □

For the remainder of this section $\phi : F(X) \longrightarrow F(X)$ is a simple immersion of a finitely generated free group $F(X)$ and such that $|\phi(x)| \geq 2$ for each $x \in X$.

Lemma 3.5. *Suppose that $|f| = |g|$ and that both $\phi(f)$, $\phi(g)$ are initial segments of a freely reduced word w in X.*

Then $f = g$.

Proof. Put $k = |f| = |g|$. Let $f = x_1 \dots x_k$ and $g = y_1 \dots y_k$ be freely reduced forms of f and g in X.

We will show by induction on $i = 1, \dots, k$ that $x_i = y_i$.

Base of induction. Let $i = 1$. Let a be the first letter of w. This implies that a is the first letter of both $\phi(x_1)$ and $\phi(y_1)$. Since ϕ is simple, we conclude that $x_1 = a = y_1$ as required.

Inductive step. Suppose that $1 < i \leq k$ and that $x_j = y_j$ for all $1 \leq j < i$.

We know that the words $\phi(x_1)\dots\phi(x_{i-1})\phi(x_i)\dots\phi(x_k)$ and $\phi(x_1)\dots\phi(x_{i-1})\phi(y_i)\dots\phi(y_k)$ are freely reduced and are initial segments of w.

Therefore the words $\phi(x_i)\dots\phi(x_k)$ and $\phi(y_i)\dots\phi(y_k)$ are initial segments of $\phi(x_1 \dots x_{i-1})^{-1}w$. Hence by the same argument as for $i = 1$ we conclude that $x_i = y_i$.

This completes the inductive step and the proof of Lemma 3.5. □

Proposition 3.6. *Let $\phi : F(X) \longrightarrow F(X)$ be a simple endomorphism of a finitely generated free group $F(X)$. Assume that $|\phi(x)| \geq 2$ for each $x \in X$. Let v be a proper initial segment of $\phi(a)$ for some $a \in X$.*

Suppose $f, h \in F(X)$ are cyclically reduced elements starting with a such that $v^{-1}fv \in \phi(F(X))$ and $v^{-1}hv \in \phi(F(X))$.

Then $[f, h] = 1$.

Proof. Note that in a free group two nontrivial elements commute if and only if some nonzero powers of them are equal. Note also that f and h are cyclically reduced. Therefore, by taking powers if necessary we may assume that $|f| = |h|$.

We will show that in fact $f = h$ (and so $[f, h] = 1$).

Let $w = x_1 \dots x_k = f$ and $w_1 = y_1 \dots y_k = h$ be freely reduced words in X representing f and h respectively. By our assumptions $x_1 = y_1 = a$.

If $k = 1$ then $f = h = a$ and there is nothing to prove. So from now on we assume that $k \geq 2$. Note also that since v is a proper initial segment of $\phi(a)$ and ϕ is simple, we have $v \notin \phi(F(X))$.

For each $z \in X \cup X^{-1}$ denote $\phi(x)$ by Z.

Then

$$W = \phi(x_1)\dots\phi(x_k) = X_1\dots X_k = \phi(f),$$
$$W_1 = \phi(y_1)\dots\phi(y_k) = Y_1\dots Y_k = \phi(h).$$

Let $f', h' \in F(X)$ be such that $v^{-1}\phi(f)v = \phi(f')$ and $v^{-1}\phi(h)v = \phi(h')$. Let $u = s_1\dots s_m$ and $u_1 = z_1\dots z_t$ be freely reduced forms of f' and h' in X accordingly. Thus

$$U = \phi(s_1)\dots\phi(s_m) = S_1\dots S_m = \phi(f'),$$
$$U_1 = \phi(z_1)\dots\phi(z_t) = Z_1\dots Z_t = \phi(h').$$

Since v is an initial segment of $A = \phi(a)$, we have $A = vr$. We also have the following equalities of freely reduced words in X:

$$rX_2\dots X_k v = S_1\dots S_m$$
$$rY_2\dots Y_k v = Z_1\dots Z_t.$$

Claim. We will show by induction on $i = 1,\dots,k$ that $X_i = Y_i$.

Base of induction. Let $i = 1$. We are given that $x_1 = a = y_1, X_1 = \phi(a) = Y_1$.

Inductive step. Let $i \geq 2, i \leq k$ and assume that $x_j = y_j, X_j = Y_j$ for $1 \leq j < i$.

Both S_1 and Z_1 have the same first letter as does r. Since ϕ is simple, this in fact means that $s_1 = z_1$ and $S_1 = Z_1$.

Case 1. Suppose $|S_1| \leq |rX_2\dots X_{i-1}| = |rY_2\dots Y_{i-1}|$.

Let $j*$ be maximal j such that both $S_1\dots S_j$ and $Z_1\dots Z_j$ are initial segments of $rX_2\dots X_{i-1} = rY_2\dots Y_{i-1}$. Note that $rX_2\dots X_k q = S_1\dots S_m$,
$rY_2\dots Y_k q = Z_1\dots Z_t$ and therefore $j < m, j < t$.

It follows from Lemma 3.5 that $S_j = Z_j, s_j = z_j$ for $j \leq j*$,. The choice of $j*$ implies that either $S_1\dots S_{j*}S_{j*+1}$ or $Z_1\dots Z_{j*}Z_{j*+1}$ is not an initial segment of $rX_2\dots X_{i-1} = rY_2\dots Y_{i-1}$. Suppose the former takes place (the other case is symmetric). Then $|S_1| + \dots + |S_{j*}| + |S_{j*+1}| > |rX_2\dots X_{i-1}| = |rY_2\dots Y_{i-1}|$.

Note that $S_1\dots S_{j*} \neq rX_2\dots X_{i-1}$ since $r \notin \phi(F)$. Thus $|S_1|+\dots+|S_{j*}| < |rX_2\dots X_{i-1}| = |rY_2\dots Y_{i-1}|$.

Let x be the $|S_1|+\dots+|S_{j*}|+1$-st letter of letter of $rX_2\dots X_{i-1} = rY_2\dots Y_{i-1}$. Then x is the first letter of both S_{j*+1} and Z_{j*+1}. Hence $S_{j*+1} = Z_{j*+1} = \phi(x), s_{j*+1} = x = z_{j*+1}$. Since $|S_1|+\dots+|S_{j*}|+|S_{j*+1}| > |rX_2\dots X_{i-1}|$, the word $S_1\dots S_{j*}S_{j*+1}$ has at least $|rX_2\dots X_{i-1}|+1$ letters. Let y be the $|rX_2\dots X_{i-1}|+1$-st letter of $S_1\dots S_{j*}S_{j*+1}$. Then y is the first letter of both X_i and Y_i.

Since ϕ is simple, this means that $X_i = \phi(y) = Y_i, x_i = y = y_i$.

Case 2. Suppose either $|S_1| > |rX_2\dots X_{i-1}| = |rY_2\dots Y_{i-1}|$.

Recall that $Z_1 = S_1$. Let y be the $|rX_2\dots X_{i-1}| + 1$-st letter of S_1. Hence x is the first letter of both X_i and Y_i. Therefore, since ϕ is simple, $x_i = x = y_i, X_i = \phi(x) = Y_i$. This completes the inductive step. The Claim is proved.

Thus $X_i = Y_i, x_i = y_i$ for $1 \leq i \leq k$ and hence $f = h$ as required.

Proposition 3.6 is proved. □

Proposition 3.7. *Let $\phi : F(X) \longrightarrow F(X)$ be a simple endomorphism of a finitely generated free group $F(X)$. Assume that $|\phi(x)| \geq 2$ for each $x \in X$. Suppose also that ϕ does not have any periodic conjugacy classes in $F(X)$.*

Then there exists $n \geq 1$ such that for any $g \in F(X)$ $g \notin \phi(F(X))$ we have

$$g^{-1}\phi^n(F(X))g \cap \phi^n(F(X)) = 1$$

Proof. Suppose the statement of Proposition 3.7 does not hold. Then there exists a sequence of elements $g_i \in F(X) - \phi(F(X))$, $i = 1, 2, \dots$ such that

$$g_i^{-1}\phi^i(F(X))g_i \cap \phi^i(F(X)) \neq 1 \tag{8}$$
$$g_i^{-1}\phi^i(f_i)g_i = \phi^i(h_i), \quad f_i \neq 1, h_i \neq 1$$

for any $i \geq 1$, $i \in \mathbf{Z}$.

Write each f_i, h_i in the form

$$f_i = a_i^{-1}b_ia_i \tag{9}$$
$$h_i = c_id_ic_i^{-1}$$

Where elements b_i, d_i are cyclically reduced in $F(X)$.

Then

$$g_i^{-1}\phi^i(a_i)^{-1}\phi^i(b_i)\phi^i(a_i)g_i = \phi^i(c_i)\phi^i(d_i)\phi^i(c_i)^{-1} \tag{10}$$

Since b_i,d_i are cyclically reduced and ϕ is simple, the elements $\phi^i(b_i)$ and $\phi^i(d_i)$ are also cyclically reduced. Since $\phi^i(b_i)$ and $\phi^i(d_i)$ are conjugate in $F(X)$, this implies that $\phi^i(d_i)$ is a cyclic permutation of $\phi^i(b_i)$ in $F(X)$.

Thus

$$\phi^i(b_i) \equiv \phi(\alpha_i)v_iw_i\phi(\beta_i) \tag{11}$$
$$\phi^i(d_i) \equiv w_i\phi(\beta_i)\phi(\alpha_i)v_i \equiv w_i\phi(\beta_i\alpha_i)v_i$$

where $v_iw_i \equiv \phi(x_i)$ for some $x_i \in X \cup X^{-1}$, $w_i \neq 1$.

Note that $|v_i| \leq \max\{|\phi(y)|$ where $y \in X\}$. Hence the set of all v_i is finite. Therefore there exists a strictly increasing infinite sequence i_k and such that

$$v_{i_k} = v \text{ for every } k \geq 1$$

<u>Case 1.</u> Suppose that $|v| > 0$, that is $v \neq 1$. Thus v is a proper initial segment of $\phi(x_{i_k})$ and so $v \notin \phi(F)$. Since ϕ is simple, this means that there is $x \in X \cup X^{-1}$ such that $x_{i_k} = x$ for any k. Note that $w_{i_k} = w$ for every k and $x \equiv vw$.

Thus

$$\phi^{i_k}(b_{i_k}) \equiv \phi(\alpha_{i_k})vw\phi(\beta_{i_k}) = \phi(\alpha_{i_k}) \cdot vw\phi(\beta_{i_k}\alpha_{i_k}) \cdot \phi(\alpha_{i_k})^{-1} = \tag{12}$$
$$= \phi(\alpha_{i_k}) \cdot \phi(x)\phi(\beta_{i_k}\alpha_{i_k}) \cdot \phi(\alpha_{i_k})^{-1} = \phi(\alpha_{i_k}) \cdot \phi(x\beta_{i_k}\alpha_{i_k}) \cdot \phi(\alpha_{i_k})^{-1},$$

Also

$$\phi^{i_k}(d_{i_k}) \equiv w\phi(\beta_{i_k}\alpha_{i_k})v \in \phi(F) \tag{13}$$

Hence $vw\phi(\beta_{i_k}\alpha_{i_k}) = \phi(x\beta_{i_k}\alpha_{i_k})$ is such that its cyclic permutation by the proper initial segment v of $\phi(x)$ is equal to a cyclically reduced word in $\phi(F)$.

Therefore by Proposition 3.6 all the elements $\phi(x\beta_{i_k}\alpha_{i_k})$ pairwise commute with each other for $k \geq 1$ and therefore generate a cyclic subgroup of $F(X)$. Hence there exists $r \in F(X)$ such that

$$vw\phi(\beta_{i_k}\alpha_{i_k}) = \phi(x\beta_{i_k}\alpha_{i_k}) = r^{n_k} \tag{14}$$

for every $k \geq 1$. This implies that

$$\phi^{i_k}(b_{i_k}) \equiv \phi(\alpha_{i_k})vw\phi(\beta_{i_k}) = \phi(\alpha_{i_k}) \cdot vw\phi(\beta_{i_k})\phi(\alpha_{i_k}) \cdot \phi(\alpha_{i_k})^{-1} = \tag{15}$$
$$= \phi(\alpha_{i_k})r^{n_k}\phi(\alpha_{i_k})^{-1}$$

By taking a subsequence of i_k and inverting r if necessary, we may assume that $n_k > 0$ for every k. Note that r^{n_k} is a cyclic permutation of a cyclically reduced word $\phi^{i_k}(b_{i_k})$ and so r is cyclically reduced itself.

By taking a subsequence of $(i_k)_k$ if necessary, we may assume that all the words b_{i_k} contain the same letter $y \in X \cup X^{-1}$. Since ϕ is simple, this means that $\phi^{i_k}(b_{i_k})$ contains a subword $\phi^{i_k}(y)$ for $k \geq 1$.

Note also that $|\phi^{i_k}(y)| \geq 2^{i_k}$ by the properties of simple immersions. Since $\phi^{i_k}(b_{i_k})$ is a cyclic permutation of a power of r for every k, this implies that $\phi^{i_k}(y)$ contains as a subword an arbitrarily large power of r.

Let N be the total number of elements of $F(X)$ of length at most $|r|$. There exists p such that $\phi^{i_k}(y)$ contains the subword r^N, that is $\phi^{i_p}(y) \equiv Ar^N B$. This implies that

$$\phi^{i_{p+1}}(y) \equiv \phi^{i_{p+1}-i_p}(A)\phi^{i_{p+1}-i_p}(r)^N \phi^{i_{p+1}-i_p}(B). \tag{16}$$

On the other hand $\phi^{i_{p+1}}(y)$ is a cyclic permutation of a positive power of r and $|\phi^{i_{p+1}}(y)| > |\phi^{i_p}(y)|$. Hence

$$\phi^{i_{p+1}}(y) \equiv \alpha r^M \beta \tag{17}$$

where $|\alpha| \leq |r|, |\beta| \leq |r|$ and $N \leq M$. Hence for every initial segment u of $\phi^{i_{p+1}}(y) \equiv \alpha r^M \beta$ there is $\epsilon \in F(X)$, $|\epsilon| \leq |r|$ such that $u\epsilon = \alpha r^s$ for some $0 \leq s \leq M$.

Thus for each $j = 0, 1, \ldots, N$ there is $\epsilon_j \in F(X)$, $s_j \leq M, s_j \geq 0$ such that $s_j \leq s_{j+1}$, $|\epsilon_j| \leq |r|$ and

$$\phi^{i_{p+1}-i_p}(A)\phi^{i_{p+1}-i_p}(r)^j \epsilon_j = \alpha r^{s_j} \tag{18}$$

By the choice of N this means that there are t, l, $0 \leq t < l \leq N$ such that $\epsilon_t = \epsilon_l = \epsilon$.

Hence

$$\phi^{i_{p+1}-i_p}(A)\phi^{i_{p+1}-i_p}(r)^t \epsilon = \alpha r^{s_t}$$
$$\phi^{i_{p+1}-i_p}(A)\phi^{i_{p+1}-i_p}(r)^l \epsilon = \alpha r^{s_l}$$

and therefore

$$\epsilon^{-1}\phi^{i_{p+1}-i_p}(r)^{l-t}\epsilon = r^{s_l - s_t} \tag{19}$$

Note that $l > t$ implies $s_l \geq s_t$, $s_l - s_t \geq 0$ by the choice of s_j. Also $i_{p+1} > i_p$ and so $i_{p+1} - i_p > 0$. Since ϕ is injective, this implies that $s_l - s_t > 0$. Thus ϕ has a periodic conjugacy class which contradicts our assumptions.

<u>Case 2.</u> Suppose that $|v| = 0, v = 1$.

Then $w_{i_k} = x_{i_k}$ and

$$\begin{aligned} \phi^{i_k}(b_{i_k}) &\equiv \phi(\alpha_{i_k})\phi(x_{i_k}\beta_{i_k}) \\ \phi^{i_k}(d_{i_k}) &\equiv \phi(x_{i_k}\beta_{i_k})\phi(\alpha_{i_k}) \end{aligned} \tag{20}$$

Hence

$$\begin{aligned} &\phi(\alpha_{i_k})^{-1}\phi^{i_k}(b_{i_k})\phi(\alpha_{i_k}) = \phi^{i_k}(d_{i_k}) \\ &\phi(\alpha_{i_k})(c_{i_k})\phi(\alpha_{i_k})^{-1}\phi^{i_k}(b_{i_k})\phi(\alpha_{i_k})\phi(\alpha_{i_k})(c_{i_k})^{-1} = \\ &= \phi(\alpha_{i_k})(c_{i_k})\phi^{i_k}(d_{i_k})\phi(\alpha_{i_k})(c_{i_k})^{-1} \end{aligned} \tag{21}$$

On the other hand

$$g_i^{-1}\phi^i(a_i)^{-1}\phi^i(b_i)\phi^i(a_i)g_i = \phi^i(c_i)\phi^i(d_i)\phi^i(c_i)^{-1} \tag{22}$$

and therefore (compare with (9), (10))

$$[\phi^{i_k}(b_{i_k}), \phi^{i_k}(a_{i_k})g_{i_k}\phi^{i_k}(c_{i_k})\phi(\alpha_{i_k})^{-1}] = 1 \tag{23}$$

Denote $\bar{g}_{i_k} = \phi^{i_k}(a_{i_k})g_{i_k}\phi^{i_k}(c_{i_k})\phi(\alpha_{i_k})^{-1}$. Since $g_{i_k} \notin \phi(F)$, we conclude that $\bar{g}_{i_k} \notin \phi(F)$. Also, we know that in a free group two nontrivial elements commute if and only if some nonzero powers of them are equal. Hence

$$\bar{g}_{i_k}^{n_k} = \phi^{i_k}(b_{i_k})^{m_k} \tag{24}$$

for some $n_k \neq 0, m_k \neq 0$. Inverting $\bar{g}_{i_k}$ if necessary, we may assume that $n_k > 0, m_k > 0$ for every k.

Recall that b_i are all cyclically reduced. Therefore $\bar{g}_{i_k}$ are cyclically reduced as well.

Put $\bar{f}_{i_k} = \bar{h}_{i_k} = b_{i_k}^{m_k}$, $\bar{b}_{i_k} = \bar{d}_{i_k} = b_{i_k}^{m_k}$ $\bar{a}_{i_k} = 1$, $\bar{c}_{i_k} = 1$ for every k. Then (compare with (10), (11)):

$$\begin{aligned} \bar{g}_{i_k}^{-1}\phi^{i_k}(\bar{a}_{i_k})^{-1}\phi^{i_k}(\bar{b}_{i_k})\phi^{i_k}(\bar{a}_{i_k})\bar{g}_{i_k} &= \phi^{i_k}(\bar{c}_{i_k})\phi^{i_k}(\bar{d}_{i_k})\phi^{i_k}(\bar{c}_{i_k})^{-1} \\ \bar{f}_{i_k} &= \bar{a}_{i_k}^{-1}\bar{b}_{i_k}\bar{a}_{i_k} \\ \bar{h}_{i_k} &= \bar{c}_{i_k}^{-1}\bar{d}_{i_k}\bar{c}_{i_k} \end{aligned} \tag{25}$$

and the words $\bar{b}_{i_k}$, $\bar{d}_{i_k}$ are cyclically reduced. Recall that

$$\phi^{i_k}(\bar{b}_{i_k}) = \phi^{i_k}(\bar{d}_{i_k}) = \phi^{i_k}(b_{i_k}^{m_k}) = \bar{g}_{i_k}^{n_k} \tag{26}$$

Hence $\bar{g}_{i_k}$ is an initial segment of $\phi^{i_k}(\bar{b}_{i_k})$ and the cyclic permutation of $\phi^{i_k}(\bar{b}_{i_k})$ by this initial segment produces $\phi^{i_k}(\bar{d}_{i_k})$:

$$\begin{aligned} \bar{g}_{i_k}^{-1}\bar{g}_{i_k}^{n_k}\bar{g}_{i_k} &= \bar{g}_{i_k}^{n_k} \\ \bar{g}_{i_k}^{-1}\phi^{i_k}(\bar{b}_{i_k})\bar{g}_{i_k} &= \phi^{i_k}(\bar{d}_{i_k}) \end{aligned} \tag{27}$$

Since $\bar{g}_{i_k} \notin \phi(F)$ is an initial segment of $\phi^{i_k}(\bar{b}_{i_k})$, we have (compare with (11)):

$$\begin{aligned} \phi^{i_k}(\bar{b}_{i_k}) &\equiv \phi(\bar{\alpha}_{i_k})\bar{v}_{i_k}\bar{w}_{i_k}\phi(\bar{\beta}_{i_k}) \\ \bar{g}_{i_k} &\equiv \phi(\bar{\alpha}_{i_k})\bar{v}_{i_k} \end{aligned} \tag{28}$$

where $\bar{v}_{i_k}\bar{w}_{i_k} \equiv \phi(\bar{x}_{i_k})$ for some $\bar{x}_{i_k} \in X \cup X^{-1}$, $\bar{w}_{i_k} \neq 1$. Moreover, since $\bar{g}_{i_k} \notin \phi(F)$, we have $|\bar{v}_{i_k}| > 0, \bar{v}_{i_k} \neq 1$ for every k. By taking a subsequence of i_k if we may assume that $\bar{v}_{i_k} = \bar{v} \neq 1$ for every k. Hence Case 1 applies. This completes the proof of Proposition 3.7. □

4. The Combination Theorem for HNN-extensions

In this section we discuss a particular case of the Combination Theorem of M.Bestvina and M.Feighn [BF92] . Namely, we will describe a set of sufficient conditions which ensure that an HNN-extension of a word-hyperbolic group is again word-hyperbolic (for definition and basic properties of word-hyperbolic groups see [Gr87], [GH90]).

Definition 4.1. Let C be a group and let $\alpha_{-1}, \alpha_1 : C \longrightarrow G$ be two monomorphisms from C into a group G. Let

$$G^* =< G, t | t^{-1}\alpha_{-1}(c)t = \alpha_1(c), c \in C > \tag{29}$$

be the HNN-extension of G corresponding to α_{-1}, α_1. We say that a *combinatorial annulus* of length $2M+1$ is a pair $\Sigma = (p, \underline{c})$ which satisfies the following requirements.

(a) p is a sequence of the form

$$p = t^{\epsilon_{-M}}, a_{-M}, t^{\epsilon_{-M+1}}, a_{-M+1}, \dots t^{\epsilon_0}, a_0, t^{\epsilon_1}, \dots t^{\epsilon_{M-1}}, a_{M-1}, t^{\epsilon_M}$$

where $a_i \in G, i = -M, \dots, M-1$ and $\epsilon_i = \pm 1$, $i = -M, \dots, M$;

(b) $\underline{c}$ is a sequence of the form

$$\underline{c} = c_{-M}, c_{-M+1}, \dots c_0, c_1, \dots c_M$$

where $c_i \in C, c_i \neq 1$ for $i = -M, \dots, M$;

(c) for every $i = -M, \dots, M-1$ we have

$$a_i^{-1}\alpha_{\epsilon_i}(c_i)a_i = \alpha_{-\epsilon_{i+1}}(c_{i+1}) \text{ in } G.$$

Remark 4.2. Suppose that Σ is an annulus as above. Then in G^* we have

$$(\overline{p})^{-1}\alpha_{-\epsilon_{-M}}(c_{-M})\overline{p} = \alpha_{\epsilon_M}(c_M),$$

where

$$\overline{p} = t^{\epsilon_{-M}}a_{-M}t^{\epsilon_{-M+1}}a_{-M+1}\dots t^{\epsilon_0}a_0t^{\epsilon_1}\dots t^{\epsilon_{M-1}}a_{M-1}t^{\epsilon_M}.$$

We will call $\overline{p}$ the *upper label* of the annulus Σ.

Definition 4.3. An annulus $\Sigma = (p, \underline{c})$ of length $2M+1$ is called *essential* if the sequence p has no "pinches", that is p has no subsequences of the form

$$t^{-1}, \alpha_{-1}(c), t, \text{ where } c \in C$$

or

$$t, \alpha_1(c), t^{-1}, \text{ where } c \in C.$$

Remark 4.4. Notice that if $\Sigma = (p, \underline{c})$ is an essential annulus then the sequence p represents an edge-path of length $2M+1$ without reversals in the Bass-Serre covering tree (see [Ser80]) corresponding to the HNN presentation of G^* and, moreover, the element $\alpha_{-\epsilon_{-M}}(c_{-M}) \in G^*$ fixes the path p pointwise.

Definition 4.5. Let G^* be as in (3). Suppose that both groups G and C are finitely generated. Fix a finite generating set $\mathcal{G}$ for G and a finite generating set $\mathcal{C}$ for C. Let $l_{\mathcal{G}}$ denote the word length induced by $\mathcal{G}$ on G and let $l_{\mathcal{C}}$ denote the word length induced by $\mathcal{C}$ on C. Let $\Sigma = (p, \underline{c})$ be an annulus of length $2M+1$.

We say that the *girth* of Σ is $l_{\mathcal{C}}(c_0)$.
We say that the *width* of Σ is $\max\{l_{\mathcal{G}}(a_i) | i = -M, \dots M-1\}$.
If $\lambda > 1$, we say that the annulus Σ is λ-hyperbolic provided

$$\lambda \cdot l_{\mathcal{C}}(c_0) \le \max\{l_{\mathcal{C}}(c_{-M}), l_{\mathcal{C}}(c_M)\}.$$

The following statement is a particular case of the main result of [BF92].

Theorem 4.6 (Combination Theorem for HNN-extensions)**.** *Let G be a word hyperbolic group with a finite generating set $\mathcal{G}$ and the corresponding word length $l_{\mathcal{G}}$. Let C be a finitely generated group with a finite generating set $\mathcal{C}$ and the corresponding word length $l_{\mathcal{C}}$. Let $\alpha_{-1}, \alpha_1 : C \longrightarrow G$ be group monomorphisms such that the subgroups $\alpha_{-1}(C)$ and $\alpha_1(C)$ are quasiconvex in G (see* [KS96], [Gr87] *for the definition and properties of quasiconvex subgroups). Let G^* be the HNN-extension*

$$G^* =< G, t | t^{-1}\alpha_{-1}(c)t = \alpha_1(c), c \in C >$$

Suppose there exist a real number $\lambda > 1$ and an integer $M > 0$ with the following properties. For any $\rho > 0$ there is $H(\rho)$ such that every essential annulus of length $2M+1$, width at most ρ and girth at least $H(\rho)$ is λ-hyperbolic.

Then G^* is word-hyperbolic.

Definition 4.7. We say that a subgroup H of a group G is *malnormal* in G if

$$g^{-1}Hg \cap H = 1 \text{ for any } g \in G, g \notin H.$$

Definition 4.8. We say that an annulus $\Sigma = (p, \underline{c})$ is *one-directed* if $\epsilon_i = \epsilon_j$ for any i, j, $-M \le i, j \le M$.

We say that an annulus $\Sigma = (p, \underline{c})$ has *one reversal* if there is k, $-M \le k < M$ such that $\epsilon_k = -\epsilon_{k+1}$ and

$$\epsilon_i = \epsilon_k \text{ for } i \le k$$
$$\epsilon_i = \epsilon_{k+1} \text{ for } i \ge k+1$$

In this situation k is called the *reversal index*.

The following lemma follows immediately from the definition of an essential annulus and that of a malnormal subgroup.

Lemma 4.9. *Let G^* be the HNN-extension of G given by presentation (29). Suppose $\alpha_{-1}(C)$ is malnormal in G. Let $\Sigma = (p, \underline{c})$ be an essential annulus for (29). Then the following holds.*

(a) *The sequence p does not contain subsequences of the form t^{-1}, a, t, where $a \in G$.*
(b) *The annulus Σ is either one-directed or has one reversal.*

Proof. It is obvious that (a) implies (b). Suppose that (a) does not hold and that p contains a subsequence t^{-1}, a, t, that is $a = a_i, t = t_i, t^{-1} = t_{i+1}$ for some i. Since Σ is essential, $a \notin \alpha_{-1}(C)$. On the other hand, by Definition 4.1 we have

$$a^{-1}\alpha_{-1}(c_i)a = \alpha_{-1}(c_{i+1})$$

and hence $a^{-1}\alpha_{-1}(C)a \cap \alpha_{-1}(C) \neq 1$. This contradicts our assumption that $\alpha_{-1}(C)$ is malnormal in G. □

Remark 4.10. Note that G is malnormal in G so Lemma 4.9 applies when $\alpha_{-1}(C) = G$.

5. Mapping tori of endomorphisms and the Combination Theorem

Let $F(X)$ be an finitely generated free group on X. Let $\phi : F(X) \longrightarrow F(X)$ be a simple immersion of $F(X)$ such that $|\phi(x)| \geq 2$ for every $x \in X$.

Put $C = F(X)$, $\alpha_{-1} = id_{F(X)}$ and $\alpha_1 = \phi$. Then the mapping torus group M_ϕ has an HNN-presentation

$$M_\phi = \langle F(X), t \,|\, t^{-1}\alpha_{-1}(c)t = \alpha_1(c), \text{ for every } c \in C\rangle \tag{30}$$

as in (29).

Let $A = \max\{|\phi(x)|, \text{ where } x \in X\}$. Then $2 \leq A$ and for any $f \in F(X)$, $k \geq 1$ we have

$$2^n \leq |\phi^k(f)| \leq A^n|f|. \tag{31}$$

Lemma 5.1. *Suppose $\Sigma = (p, \underline{c})$ is an annulus of widths at most ρ with respect to (30). (We put the standard word metric on both the base of the HNN-extension $F(X)$ and on the edge group $C = F(X)$.) Suppose further that in Σ we have $t_i = t_{i+1} = \cdots = t_{i+k} = t$ for some $i, -M \leq i < M$ and some $k \geq 1$.*

Then

$$2^k|c_i| - 2\rho\frac{A^k - 1}{A - 1} \leq |c_{i+k}| \leq A^k|c_i| + 2\rho\frac{A^k - 1}{A - 1}. \tag{32}$$

Proof. The definition of the annulus implies that $c_{j+1} = a_j^{-1}\phi(c_j)a_j$ for $i \leq j < i + k$. Hence

$$c_{i+k} = y^{-1}\phi^k(c_i)y \tag{33}$$

where $y = \phi^{k-1}(a_i)\phi^{k-2}(a_{i+1})\dots\phi(a_{i+k-2})a_{i+k-1}$. Since Σ has width at most ρ, $|a_j| \leq \rho$ for each j. Thus by (32) we have

$$|y| \leq A^{k-1}\rho + A^{k-2}\rho + \dots A\rho + \rho = \rho\frac{A^k - 1}{A - 1}.$$

Therefore

$$|c_{i+k}| \leq |\phi^k(c_i)| + 2|y| \leq A^k|c^i| + 2\rho\frac{A^k - 1}{A - 1}$$

$$|c_{i+k}| \geq |\phi^k(c_i)| - 2|y| \geq 2^k|c^i| - 2\rho\frac{A^k - 1}{A - 1}$$

Thus Lemma 5.1 is proved. □

Lemma 5.2. *Suppose $\Sigma = (p, \underline{c})$ is an annulus of widths at most ρ with respect to* (30)*. Suppose further that in Σ we have $t_0 = t_1 = \cdots = t_i = t$, $t_{i+1} = t_{i+2} = \cdots = t_{i+j} = t^{-1}$ for some $i, j \geq 1$.*

Then

$$|c_{i+j}| \geq \frac{2^i}{A^j}|c_0| - 2\rho\frac{A^i + A^j - 2}{A^j(A-1)} \tag{34}$$

Proof. Lemma 5.1 applied to Σ implies that

$$|c_i| \geq 2^i|c_0| - 2\rho\frac{A^i - 1}{A - 1} \tag{35}$$

On the other hand, Lemma 5.1 applied to the mirror image of Σ implies that

$$|c_i| \leq A^j|c_{i+j}| + 2\rho\frac{A^j - 1}{A - 1} \tag{36}$$

$$|c_{i+j}| \geq \frac{|c_i|}{A^j} - 2\rho\frac{A^j - 1}{A^j(A-1)} \tag{37}$$

Hence by (35) and (37)

$$|c_{i+j}| \geq \frac{2^i}{A^j}|c_0| - 2\rho\frac{A^i + A^j - 2}{A^j(A-1)}$$

as required. Lemma 5.2 is proved. □

Lemma 5.3. *Let Σ be an essential annulus with respect to the HNN-presentation* (30) *which has exactly one reversal. Suppose this reversal has reversal index k and $0 \leq k < M$. Then there exists $g \in F(X) - \phi(F(X))$ such that*

$$g^{-1}\phi^{M-k}(F(X))g \cap \phi^{M-k}(F(X)) \neq 1 \tag{38}$$

Proof. Note that $\alpha_{-1}(C) = F(X)$ is malnormal in $F(X)$. Therefore by Lemma 4.9 the reversal in Σ has the form t, a, t^{-1} where $a \notin \phi(F(X))$. This means that $t_k = t, a_k = a, t_{k+1} = t^{-1}$ and

$$t_i = t \text{ for } -M \leq i \leq k$$

$$t_i = t^{-1} \text{ for } k < i \leq M.$$

Now the definition of an annulus immediately implies Lemma 5.3. □

Proposition 5.4. *Let $F(X)$ be an finitely generated free group on X. Let $\phi : F(X) \longrightarrow F(X)$ be a simple immersion of $F(X)$. Suppose that ϕ has no periodic conjugacy classes.*

Then the mapping torus group M_ϕ is word-hyperbolic.

Proof. Note first that $|\phi(x)| \geq 2$ for every $x \in X$. Indeed, if $|\phi(x)| = 1$ for some $x \in X$ then by simplicity of ϕ $\phi(x) = x$ and ϕ has a periodic conjugacy class contrary to our assumptions. Put $C = F(X)$, $\alpha_{-1} = id_{F(X)}$ and $\alpha_1 = \phi$. Then the mapping torus group M_ϕ has an HNN-presentation as in (30).

Since the group $F(X)$ is a free group of finite rank, every finitely generated subgroup of $F(X)$ is quasiconvex in $F(X)$ [Sho91]. In particular, $\alpha_{-1}(C) = F(X)$ and $\alpha_1(C) = \phi(F(X))$ are quasiconvex in $F(X)$. Thus the basic assumption of Theorem 4.6 is satisfied.

Put $A = \max\{|\phi(x)|, x \in X\}$ (so that $A \geq 2$). By Proposition 3.7 there exists $k \geq 1$ such that for any $g \in F(X) - \phi(F(X))$ and any $j \geq k$

$$g^{-1}\phi^j(F(X))g \cap \phi^j(F(X)) = 1$$

Choose n so that $n \geq 10k$, $2^n \geq 10A^k$ and $2^n > 10$. Put $M = n + k$. Put $\alpha_M = 4\frac{A^M}{A-1}$.

For every $\rho > 0$ put

$$H(\rho) = \frac{\rho\alpha_M}{8} \tag{39}$$

Let $\rho \geq 1$ be an integer. Let $\Sigma = (p, \underline{c})$ be an essential annulus of length $2M+1$, width at most ρ and girth at least $H = H(\rho)$.

By taking a mirror image of Σ if necessary we may assume that $t_0 = t$. By Lemma 4.9 either Σ is one-directed or it has exactly one reversal of the type t, a, t^{-1}, $a \notin \phi(F)$. There are two cases two consider.

Case 1. Suppose first that there are no reversals in Σ for positive indices, that is $t_i = t$ for $0 \leq i \leq M$. In this case by Lemma 5.1 we have

$$|c_M| \geq 2^M |c_0| - 2\rho \frac{A^M - 1}{A-1} \geq 2^M |c_0| - \rho\alpha_M \geq 10|c_0| - \rho\alpha_M. \tag{40}$$

The annulus Σ has girth at least H, that is $|c_0| \geq H$. By the choice of $H = H(\rho)$ this an equation (40) imply that $|c_M| \geq 2|c_0|$, that is the annulus Σ is 2-hyperbolic.

Case 2. Suppose now that Σ has a reversal for some positive index, that is there is $j \geq 0$ such that

$$t_i = t, 0 \leq i \leq j$$
$$t_i = t^{-1}, j+1 \leq i \leq M$$
$$a = a_j \notin \phi(F)$$

Then by Lemma 5.3 there is $g \in F - \phi(F)$ such that $g^{-1}\phi^{M-j}(F) g \cap \phi^{M-j}(F) \neq 1$. By the choice of k this implies that $M - j < k$. Put $s = M - j$. Thus $s < k$, $j = M - s = n + k - s$ and hence $j \geq n$. Then by Lemma 5.2 we have

$$\begin{aligned} |c_M| \geq & \frac{2^j}{A^s}|c_0| - 2\rho \frac{A^j + A^s - 2}{A^s(A-1)} \geq \frac{2^n}{A^s}|c_0| - 2\rho \frac{2A^M}{(A-1)} \geq \\ \geq & \frac{2^n}{A^k}|c_0| - \rho\alpha_M \geq 10|c_0| - \rho\alpha_M \end{aligned} \tag{41}$$

Since Σ has girth at least H, $|c_0| \geq H = H(\rho)$. Together with the definition of $H(\rho)$ this fact and inequality (41) imply that $|c_M| \geq 2|c_0|$. Thus the annulus Σ is 2-hyperbolic.

We have established that any essential $(2M+1)$ annulus of width at most ρ and girth at least $H(\rho)$ is 2-hyperbolic by the Combination Theorem (Theorem 4.6) this implies that M_ϕ is word-hyperbolic. □

The above statement almost immediately implies the main result of this paper.

Theorem 5.5 (c.f. Theorem A)**.** *Let $F(X)$ be a free group of finite rank on X. Let $\phi : F(X) \longrightarrow F(X)$ be an immersion of $F(X)$. Then the following conditions are equivalent:*

(a) *The mapping torus group M_ϕ is word-hyperbolic.*
(b) *The mapping torus group M_ϕ contains no Baumslag-Solitar subgroups of the type $B(1,p)$, $p \geq 1$.*
(c) *The endomorphism ϕ has no nontrivial periodic conjugacy classes in $F(X)$.*

Proof. It is obvious that (a) implies (b). Lemma 2.3 shows that (b) implies (c).

We will show that (c) implies (a). Suppose that ϕ has no periodic conjugacy classes. Then every positive power of ϕ also has no periodic conjugacy classes.

Put $k = (2n)!$ where n is the number of elements in X. By Lemma 3.3 ϕ^k is simple. By the above observation ϕ^k has no periodic conjugacy classes. Hence by Proposition 5.4 the group M_{ϕ^k} is word-hyperbolic. However, by Lemma 2.2 M_{ϕ^k} has finite index in M_ϕ. Hence M_ϕ is word-hyperbolic as well.

Theorem 5.5 is proved. □

Corollary 5.6 (c.f. Corollary B)**.** *There is a uniform algorithm which, given an immersion ϕ of a finitely generated free group $F(X)$, decides if the mapping torus group M_ϕ is word-hyperbolic.*

Proof. We will offer the most straightforward algorithm, although there are much more efficient approaches.

By the results of M.Gromov [Gr87] (see also the work of P.Papasoglu [Pa96]) there's a uniform algorithm which, given a finite group presentation, will eventually stop if the group defined by the presentation is word-hyperbolic and will run forever otherwise. We will call this algorithm $\mathcal{A}$.

We now describe the algorithm $\mathcal{B}$ which works as follows. Let ϕ be an immersion of a finitely generated free group $F(X)$ given by its action on the generators $x \in X$. Enumerate all tuples (f, k, p) where $f \in F(X)$, $k, p \in \mathbb{Z}$, $k \geq 1$, $p > 0$. For each such tuple check if the elements $\phi^k(f)$ and f^p are conjugate in $F(X)$. Terminate the algorithm if the answer is "yes". It is clear that for any such ϕ algorithm $\mathcal{B}$ will eventually terminate if ϕ has a periodic conjugacy class and will run forever otherwise.

Now suppose we are given an immersion ϕ of a finitely generated free group $F(X)$. We make the finite presentation for the mapping torus group M_ϕ. Now run algorithms $\mathcal{A}$ (applied to this presentation of M_ϕ) and $\mathcal{B}$ in parallel.

It follows from Theorem 5.5 that eventually either algorithm $\mathcal{A}$ will stop (in which case M_ϕ is word-hyperbolic) or algorithm $\mathcal{B}$ will stop (in which case M_ϕ is not word-hyperbolic).

□

Corollary 5.7 (c.f. Corollary C)**.** *Let $F(X)$ be a free group of finite rank on X. Let $\phi : F(X) \longrightarrow F(X)$ be a simple immersion such that $|\phi(x)| \geq 2$ for every $x \in X$.*

Then the isoperimetric function of M_ϕ is either linear or exponential.

Proof. Suppose that the isoperimetric function of M_ϕ is not linear, that is M_ϕ is not word-hyperbolic. Then by Theorem 5.5 ϕ has a nontrivial periodic conjugacy class. Thus there is $f \in F(X) - 1$, $k \geq 1$, $p > 0$ and $h \in F(X)$ such that $\phi^k(f) = h^{-1}f^p h$. We will first show that $p \geq 2$. In fact, suppose $p = 1$, that is $\phi^k(f) = h^{-1}fh$ and $||\phi^k(f)|| = ||f||$. However Proposition 3.4 shows that $||\phi^k(f)|| \geq 2^k||f|| \geq 2||f||$ which gives a contradiction.

Put $H = \langle X, t^k\rangle = \langle X, t^k h^{-1}\rangle$ and $s = t^k h^{-1}$. By Lemma 2.2 the subgroup $H = \langle t^k, X\rangle$ has finite index in M_ϕ and so the Dehn functions of H and M_ϕ are equivalent. Thus it suffices to show that the Dehn function of H is strictly exponential. Also, Lemma 2.2 implies that the group $H = \langle t^k, X\rangle = \langle t^k h^{-1}, X\rangle = \langle s, X\rangle$ is canonically isomorphic to M_θ where $\theta : F(X) \longrightarrow F(X)$ is defined as $\theta(x) = h\phi^k(x)h^{-1}$ for every $x \in X$. Since $\phi^k(f) = h^{-1}f^p h$, we have

$$\theta(f) = f^p, \;\; s^{-1}fs = f^p \tag{42}$$

Proposition 3.4 implies that θ increases the cyclically reduced length of every element by a factor of at least two. This means that θ has trivial stable image, that is $\cap_{n\geq 1}\theta^n(F(X)) = 1$. Hence there exists $b \notin \theta(F)$ and $l \geq 0$ such that $\theta^l(b) = f$. Furthermore, there is $a \notin \theta(F)$ such that a is not a proper power and $b = a^j$ for $j \geq 1$. Thus $\theta(f) = f^p$ implies $\theta(b) = b^p$ and $\theta(a^j) = a^{jp}$. Hence by commutative transitivity of free groups $\theta(a) = a^j$.

Thus $a \notin \theta(F(X)$ is not a proper power and $\theta(a) = a^p$, $p \geq 2$. Recall also that the group $H = \langle s, X\rangle$ has the following presentation on s, X:

$$\langle s, X \,|\, s^{-1}xs = \theta(x), x \in X\rangle \tag{43}$$

It is easy to see that for any $n \geq 1$ we have $[s^{-n}as^n, a] = 1$ in H.

Using the method of s-corridors, the same argument as one employed by M.Bridson and S.Gersten in the proof of Theorem 3.1 of [BG96] easily shows that a reduced H-diagram (with respect to presentation (43)) corresponding to the relation $[s^{-n}as^n, a] = 1$ is unique and has area exponential in n. On the other hand the word $[s^{-n}as^n, a]$ has length $4n + 4|a|$ in s, X, which is linear in n. Hence the Dehn function of H is at least exponential.

On the other hand $H = M_\theta$ is an HNN-extension of a free group of finite rank with finitely generated associated subgroups. Therefore by the result of [BGSS] H is asynchronously automatic. Hence H has at most exponential isoperimetric function.

Thus the Dehn function of H is strictly exponential and Corollary 5.7 is proved.

□

Remark 5.8. The same argument as in the proof of Corollary 5.7 shows that if for an arbitrary injective endomorphism ϕ of $F(X)$ the group M_ϕ contains a subgroup $B(1,p)$, $p > 1$ then M_ϕ has exponential isoperimetric inequality.

Suppose that ϕ is simple and M_ϕ contains some subgroups isomorphic to $B(1,1) \cong \mathbb{Z} \times \mathbb{Z}$ but does not contain subgroups isomorphic to $B(1,p)$, $p > 1$. (This implies that $\phi(x) = x$ for some $x \in X$.) It seems plausible that in this situation M_ϕ has quadratic isoperimetric inequality and is automatic. However, at the moment we don't have a proof of this fact.

6. Open problems

The collection of mapping tori of injective endomorphisms of free groups is a fascinating class of groups with many interesting open problems.

We will list some of them here omitting the justification for raising these problems (which in most cases is self-explanatory).

Let ϕ be an injective and non-surjective endomorphism of a finitely generated free group $F(X)$.

Problem 6.1. Is the group M_ϕ residually finite? Is the group M_ϕ linear? (It is not hard to see that M_ϕ is not subgroup separable or LERF. On the other hand, if ϕ is an automorphism, then M_ϕ is free-by-cyclic and residually finite [Ba71].)

Problem 6.2. Classify mapping tori groups up-to isomorphism. Classify mapping tori groups up-to quasi-isometry.

Problem 6.3. Is it true that M_ϕ is word-hyperbolic if and only if it does not contain Baumslag-Solitar subgroups? Does there exists a uniform (on ϕ) algorithm which decides whether or not the group M_ϕ is word-hyperbolic?

Problem 6.4. What kind of isoperimetric functions can M_ϕ have? (In all understood examples the isoperimetric function is linear, quadratic of exponential. Also, the groups M_ϕ are asynchronously automatic and so their isoperimetric functions are at most exponential.)

Problem 6.5. Does M_ϕ have solvable membership problem with respect to finitely generated subgroups? (Feighn and Handel proved that finitely generated subgroups of M_ϕ are finitely presentable).

Problem 6.6. Assuming M_ϕ is word-hyperbolic, when is $Out(M_\phi)$ infinite? That is to say, when is the essential JSJ-decomposition of a word-hyperbolic M_ϕ trivial?

If $X = X_1 \sqcup X_2$ and $\phi(X_1) \subseteq X_1, \phi(X_2) \subseteq F(X_2)$ then $M_\phi = M_{\phi_1} *_{t=t} M_{\phi_2}$ and $Out(M_\phi)$ is infinite. Here $\phi_i = \phi|_{F(X_i)}$, $i = 1, 2$.

Problem 6.7. Suppose M_ϕ is word-hyperbolic. It is known by a result of M.Mitra [Mi98] that the inclusion $i : F(X) \longrightarrow M_\phi$ extends continuously to a *Cannon-Thurston map* $\hat{i} : \partial F(X) \longrightarrow \partial M_\phi$. What kind of a map is $\hat{i}$? Is it finite-to-one (as is the case for automorphisms)? Does the ending lamination theorem hold for M_ϕ? (It does if ϕ is an automorphism [Mi97]).

Problem 6.8. The map $\phi : F(X) \longrightarrow F(X)$ always extends to a continuous self-embedding $\hat{\phi} : \partial F(X) \longrightarrow \partial F(X)$. Investigate the possible dynamics of the map $\hat{\phi}$ and try to classify injective endomorphisms in terms of their dynamics. (This dynamics is particularly simple for immersions).

Problem 6.9. Is the membership problem with respect to finitely generated subgroups of M_ϕ solvable? A possibly simpler question: for a finitely generated subgroup H of $F(X)$ is the subgroup $\langle \bigcup_{n>0} \phi^n(H) \rangle$ a recursive subset of $F(X)$?

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