Corrigendum to “Spectral rigidity of automorphic orbits in free groups”

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Lemma 5.1 in our paper [6] says that every infinite normal subgroup of $\mathrm{Out}(F_N)$ contains a fully irreducible element; this lemma was substantively used in the proof of the main result, Theorem A in [6]. Our proof of Lemma 5.1 in [6] relied on a subgroup classification result of Handel-Mosher [9], originally stated in [9] for arbitrary subgroups $H \le \mathrm{Out}(F_N)$. It subsequently turned out (see p. 1 in [10]) that the proof of the Handel-Mosher theorem needs the assumption that H be finitely generated. Here we provide an alternative proof of Lemma 5.1 from [6], which uses the corrected version of the Handel-Mosher theorem and relies on the 0-acylindricity of the action of $\mathrm{Out}(F_N)$ on the free factor complex (due to Bestvina-Mann-Reynolds).

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1 Introduction

The purpose of this note is to correct a gap in our paper [6]. The proof of the main result, Theorem A, of [6], substantively relies on Theorem 1.1 of Handel and Mosher [9] about classification of subgroups of $\mathrm{Out}(F_N)$.

Originally Theorem 1.1 was stated in [9] for arbitrary subgroups $H \le \mathrm{Out}(F_N)$, and we applied that statement in [6]. After our paper [6] was published, we were informed that the proof of Theorem 1.1 in [9] only goes through under the additional assumption that $H \le \mathrm{Out}(F_N)$ be finitely generated; see the footnote at p.1 of [10].

The specific use of Theorem 1.1 of [9] in [6] occurs in the proof of Lemma 5.1 in [6]. This proof no longer works, when the Handel-Mosher result is replaced by its finitely generated version. This situation has created a gap in the proof of Lemma 5.1 given in [6].

In this corrigendum we fix this gap and provide an alternative proof of Lemma 5.1. Thus Theorem A in [6] and all the other results proved there remain valid in their original form. Lemma 5.1 in [6] stated the following:

Proposition 1.1 *Let $N \geq 2$ and let $H \leq Out(F_N)$ be an infinite normal subgroup. Then H contains some fully irreducible element ϕ.*

Note that for $N \geq 3$ every nontrivial normal subgroup of $\mathrm{Out}(F_N)$ is infinite, but $\mathrm{Out}(F_2)$ does possess a finite nontrivial normal subgroup (namely the center of $\mathrm{Out}(F_2)$, which is cyclic of order 2). Recall also that an element $\phi \in \mathrm{Out}(F_N)$ is called *fully irreducible* or *iwip* (for "irreducible with irreducible powers") if no positive power of ϕ preserves the conjugacy class of a proper free factor of F_N.

The original formulation of Theorem 1.1 in [9] said that for an arbitrary subgroup $H \leq \mathrm{Out}(F_N)$, either H contains a fully irreducible element or H has a subgroup of finite index H_0 such that H_0 preserves the conjugacy class of some proper free factor of F_N. As noted above, it turns out that the proof of Theorem 1.1 in [9] only goes through under the additional assumption that H be finitely generated.

The new proof of Lemma 5.1 of [6], presented here, is quite different from our original argument in [6], although the proof still relies on the corrected finitely generated version of the Handel-Mosher subgroup classification theorem. Another key ingredient in this new argument is the proof, due to Bestvina, Mann and Reynolds, of 0-acylindricity of the $\mathrm{Out}(F_N)$ action on the free factor complex $\mathcal{FF}_N$.

The proof of 0-acylindricity was communicated to us by Bestvina and Reynolds. Since this proof does not appear anywhere in the literature, we include it here for completeness; see Proposition 2.2 below.

We are grateful to Ric Wade for bringing to our attention the issue with the original formulation of Theorem 1.1 of [9]. We are also grateful to Mladen Bestvina and Patrick Reynolds for explaining to us the argument for establishing 0-acylindricity of $\mathcal{FF}_N$. Finally, we thank Martin Lustig for very helpful discussions regarding the train track theory.

2 0-Acylindricity

We will use the terminology and notations from [6]. In particular, cv_N denotes the (unprojectivized) Outer space, CV_N denotes the projectivized Outer space, $\overline{\mathrm{cv}}_N$ denotes the closure of cv_N in the hyperbolic length function topology, and $\overline{\mathrm{CV}}_N$ denotes the projectvization of $\overline{\mathrm{cv}}_N$, so that $\overline{\mathrm{CV}}_N$ is the standard compactification of CV_N.

2.1 Stabilizers of free factors and reducing systems

Following [20, 5], if $T \in \overline{\mathrm{cv}}_N$ and $A \le F_N$ is a proper free factor of F_N, we say that A *reduces* T if there exists an A-invariant subtree T' of T such that A acts on T' with dense orbits (the subtree T' is allowed to consist of a single point). For $T \in \overline{\mathrm{cv}}_N$ denote by $\mathcal{R}(T)$ the set of all proper free factors of F_N which reduce T. Note that in many cases $\mathcal{R}(T)$ is the empty set.

Lemma 2.1 below is a key step in proving Proposition 2.2 establishing 0-acylindricity of the action of $Out(F_N)$ on the free factor graph. The proof of Lemma 2.1 relies on the use of geodesic currents and algebraic laminations, and is due to Patrick Reynolds. Lemma 2.1 could be replaced by a related statement, still sufficient to derive Proposition 2.2, and based the use of relative train tracks. This alternative argument is due to Brian Mann and was communicated to us by Bestvina.

If a tree $T \in \overline{\mathrm{cv}}_N$ does not have dense F_N-orbits, then it is known (see [7, 20, 14]) that T canonically decomposes as a "graph of actions". In this case there exists an F_N-equivariant distance non-increasing map $f\colon T \to Y$ for some very small simplicial metric tree $Y \in \overline{\mathrm{cv}}_N$ such for every vertex v of Y the stabilizer $Stab_{F_N}(v)$ acts with dense orbits on some $Stab_{F_N}(v)$-invariant subtree T_v of T (where T_v may be a single point). Moreover, the tree Y is obtained from T be collapsing each T_v to a point, where v varies over all vertices of Y. In this situation we will say that $Y \in \mathrm{cv}_N$ is the *simplicial tree associated* to T. Note that if $T \in \mathrm{cv}_N$ then $Y = T$.

Lemma 2.1 *Let A be a proper free factor of F_N and let $h_n \in Out(F_N)$ be an infinite sequence of distinct elements of $Out(F_N)$ such that $h_n([A]) = [A]$ for all $n \ge 1$. Let $T_0 \in \mathrm{cv}_N$ and $T \in \overline{\mathrm{cv}}_N$ be such that $[T_0]h_n \to [T]$ in $\overline{CV}_N$ as $n \to \infty$. Then:*

(1) *If T has dense F_N-orbits then there exists a nontrivial free factor A' of A such that A' reduces T.*

(2) *If T does not have dense F_N-orbits and $Y \in \overline{\mathrm{cv}}_N$ is the associated simplicial tree, then some nontrivial free factor A' of A reduces Y.*

Proof There exist $c_n \ge 0$ such that $\lim_{n\to\infty} c_n T_0 h_n = T$ in $\overline{\mathrm{cv}}_N$. Since the elements h_n are distinct and the action of $\mathrm{Out}(F_N)$ on CV_N is properly discontinuous, it follows that $T \in \overline{\mathrm{cv}}_N \setminus \mathrm{cv}_N$ and that $\lim_{n\to\infty} c_n = 0$. For every $n \ge 1$ choose a representative $\beta_n \in \mathrm{Aut}(F_N)$ of the outer automorphism h_n such that $\beta_n(A) = A$.

Choose a nontrivial element $a \in A$ and put $a_n = \beta_n^{-1}(a) \in A$. Then

$$c_n||a_n||_{T_0h_n} = ||a_n||_{c_nT_0h_n} = ||a_n||_{c_nT_0\beta_n} = c_n||\beta_n(a_n)||_{T_0} = c_n||a||_{T_0} \to_{n\to\infty} 0.$$

In $\mathbb{P}\mathrm{Curr}(F_N)$ we have $\lim_{n\to\infty} \frac{1}{||a_n||_{T_0}}\eta_{a_n} = \mu \neq 0$. Here, for a nontrivial element $g \in F_N$, $\eta_g \in \mathrm{Curr}(F_N)$ is the "counting current" associated to g [12].

Since $T_0 \in \mathrm{cv}_N$, there exists $c > 0$ such that $||g||_{T_0} \geq c > 0$ for all $g \in F_N$. Hence $||a_n||_{T_0} \geq c$ and therefore

$$\langle T, \mu\rangle = \lim_{n\to\infty} \langle c_n T_0 h_n, \frac{1}{||a_n||_{T_0}}\eta_{a_n}\rangle = \lim_{n\to\infty} \frac{c_n}{||a_n||_{T_0}}\langle T_0 h_n, \eta_{a_n}\rangle =$$

$$\lim_{n\to\infty} \frac{c_n}{||a_n||_{T_0}}||a_n||_{T_0 h_n} = \lim_{n\to\infty} \frac{c_n}{||a_n||_{T_0}}||a||_{T_0} = 0.$$

Here $\langle\ ,\ \rangle\colon \overline{\mathrm{cv}}_N \times \mathit{Curr}(F_N) \to \mathbb{R}$ is the continuous "geometric intersection form" constructed in [13].

Since $\langle T, \mu\rangle = 0$, by the main result of [14] we have $\mathit{supp}(\mu) \subseteq L(T)$, where $\mathit{supp}(\mu)$ is the support of μ and $L(T)$ is the dual algebraic lamination of T (see [7] for background about dual algebraic laminations associated to elements of $\overline{\mathrm{cv}}_N$). Since $a_n \in A$ for all $n \geq 1$, the construction of μ implies that there is a leaf of $L(T)$ that is carried by A.

If T has dense F_N-orbits, then by Corollary 6.7 of [20], there exists a nontrivial free factor A' of A such that A' reduces T, and part (1) of the lemma holds.

Suppose now that T does not have dense F_N-orbits, and let $Y \in \overline{\mathrm{cv}}_N$ be the associated simplicial tree for T.

Then by Lemma 10.2 of [14], we have $L(T) \subseteq L(Y)$. Since the factor A carries a leaf of $L(T)$, it follows that A also carries a leaf ℓ of $L(Y)$. The description of the dual lamination of a very small simplicial tree, given in Lemma 8.2 of [14], then implies that A contains some nontrivial element acting elliptically on Y. Namely, consider a free basis X of F_N such that X contains as a subset a free basis X' of A. Lemma 8.2 of [14] now implies that for some vertex group U of T' and for the Stallings core subgroup graph Δ_U [15] (with oriented edges labelled by elements of $X^{\pm 1}$) representing the conjugacy class of U, there exists a bi-infinite reduced edge-path γ in Δ_U corresponding to the leaf ℓ of $L(Y)$. The fact that ℓ is carried by A means that all the edges of γ are labelled by elements of $(X')^{\pm 1}$. Since γ is an infinite reduced path, we can find an immersed circuit as a subpath of γ. Then the label a' of this circuit is a nontrivial element of A whose conjugate belongs to U and thus a' acts elliptically on Y.

If A fixes a point of Y, then A reduces Y and we are done. Otherwise consider the minimal A-invariant subtree Y_A of Y. Then the quotient graph of groups $\mathbb{A} = Y_A//A$ gives a nontrivial very small splitting of A with at least one nontrivial vertex group. The general structural result (Lemma 4.1 of [2]) about very small simplicial splittings of free groups then implies that there exists a nontrivial free factor A' of A such that a conjugate of A' is contained in some vertex group of $\mathbb{A}$. Then A' fixes a vertex of the tree Y_A and therefore A' reduces Y, as required.

□

2.2 0-acylindricity of the free factor graph

A simplicial isometric action of a group G on a connected simplicial graph X, endowed with the simplicial metric d, is called 0*-acylindrical* if there exist constants $D, m \geq 1$ such that for any vertices x, y of X with $d(x, y) \geq D$, the set $Stab_G(x) \cap Stab_G(y) = \{g \in G | gx = x, gy = y\}$ has cardinality $\leq m$. See [19] for the background on acylindrical actions.

For $N \geq 2$ let $\mathcal{FF}_\mathcal{N}$ be the free factor graph for F_N. The vertices of $\mathcal{FF}_\mathcal{N}$ are the conjugacy classes $[A]$ of proper free factors A of F_N. For $N \geq 3$ the adjacency of vertices in $\mathcal{F}_N$ corresponds to containment: two distinct vertices $[A]$ and $[B]$ are adjacent if there exist representatives A of $[A]$ and B of $[B]$ such that $A \leq B$ or $B \leq A$. For $N = 2$ the definition of adjacency is somewhat different, see [3] for details. It is known that for $N \geq 2$ the graph $\mathcal{FF}_N$ is connected and Gromov-hyperbolic [3, 16, 11]. There is a natural action of $\mathrm{Out}(F_N)$ on $\mathcal{FF}_N$ by simplicial isometries.

We denote the simplicial metric on $\mathcal{FF}_N$ by d.

Proposition 2.2 (0-acylindricity of the free factor complex) *There exists a constant $M \geq 1$ with the following property:*

If $N \geq 2$ and if A, B are proper free factors of F_N such that $d([A],[B]) > M$ then the set

$$Stab_{\mathrm{Out}(F_N)}([A]) \cap Stab_{\mathrm{Out}(F_N)}([B])$$

is finite and has cardinality $\leq N!\ 2^N$. Thus the action of Out(F_N) on $\mathcal{FF}_N$ is 0-acylindrical.

Proof Corollary 5.3 of [5] implies that there exists a constant $C > 0$ (independent of the rank N of F_N) such that if $T \in \overline{\mathrm{cv}}_N$ admits a reducing factor then the set $\mathcal{R}(T)$ of all reducing factors for T has diameter $\leq C$ in $\mathcal{FF}_N$. Take $M = C + 2$. Let A, B be proper free factors of F_N such that $d([A],[B]) > M$. Put $H := Stab_{\mathrm{Out}(F_N)}([A]) \cap Stab_{\mathrm{Out}(F_N)}([B])$. We claim that H is finite.

Indeed, suppose not, and H is infinite. Since $\mathrm{Out}(F_N)$ is virtually torsion-free, it follows that there exists an element $h \in H$ of infinite order. Let $T_0 \in \mathrm{cv}_N$ be arbitrary and let $T \in \overline{\mathrm{cv}}_N$ be such that $\lim_{i\to\infty}[T_0 h^{n_i}] = [T]$ for some subsequence $n_i \to \infty$.

Since $h[A] = [A]$ and $h[B] = [B]$, by Lemma 2.1 there exist a tree $S \in \overline{\mathrm{cv}}_N$ and nontrivial free factors A' of A and B' of B such that A' and B' both reduce S. Namely, we can take $S = T$ if T has dense F_N orbits and otherwise we can take $S = Y$ where $Y \in \overline{\mathrm{cv}}_N$ is the simplicial tree associated to T.

Note that if $N = 2$ then $A' = A$, $B' = B$, and the factors A, B are infinite cyclic.

Then $d([A'],[A]) \le 1$, $d([B'],[B]) \le 1$ and therefore $d([A'],[B']) > M - 2 = C$. Thus the set of reducing factors for S has diameter $> C$ in $\mathcal{FF}_N$, which contradicts the choice of C.

Thus $H \le \mathrm{Out}(F_N)$ is finite. By a result of Wang and Zimmermann [21], it follows that $|H| \le N!\, 2^N$.

□

3 The proof of Proposition 1.1

We can now recover Lemma 5.1 of [6], which is stated as Proposition 1.1 above.

Proof of Proposition 1.1 Suppose that $H \le Out(F_N)$ is an infinite normal subgroup but that H does not contain a fully irreducible element. Since $\mathrm{Out}(F_N)$ is virtually torsion-free and H is infinite, it follows that H contains an element ϕ of infinite order.

By assumption on H, ϕ is not fully irreducible and hence, after replacing ϕ by a positive power, ϕ fixes the conjugacy class $[A]$ of a proper free factor A of F_N.

Now let $M \ge 1$ be the 0-acylindricity constant provided by Proposition 2.2. We choose a fully irreducible $\theta \in \mathrm{Out}(F_N)$ and look at the conjugates $\alpha_n = \theta^n \phi \theta^{-n}$. Note that α_n fixes the conjugacy class $\theta^n[A]$. Since H is normal, we have $\alpha_n \in H$, and thus the subgroup $L_n = \langle \phi, \alpha_n \rangle$ is contained in H. The subgroup L_n of $\mathrm{Out}(F_N)$ is finitely generated so the (corrected) Handel-Mosher subgroup classification theorem [9, 10] does apply to L_n.

Since we assumed that H contains no fully irreducible elements, L_n must contain a subgroup K_n of finite index which preserves a vertex $[B_n]$ of the free factor complex. Hence some positive powers of ϕ and of α_n preserve $[B_n]$.

Since θ is fully irreducible, we have $d([A], \theta^n[A]) \to \infty$ as $n \to \infty$ (see [4, 13]). We choose $n \geq 1$ big enough so that $d([A], \theta^n[A]) > 2M + 1$.

Then either $d([A], [B_n]) > M$ or $d([B_n], \theta^n[A]) > M$.

In the first case we get that some positive power ϕ^i of ϕ fixes both the vertices $[A]$ and $[B_n]$ of $\mathcal{FF}_N$, so that ϕ^i belongs to the intersection of their stabilizers. This contradicts 0-acylindricity since ϕ has infinite order.

In the second case for some $i > 0$ the element $\alpha_n^i = \theta^n \phi^i \theta^{-n} \in H$ fixes both $[B_n]$ and $\theta^n[A]$. Thus α_n^i belongs to the intersection of the stabilizers of $[B_n]$ and $\theta^n[A]$, which again contradicts 0-acylindricity, since α_n has infinite order. □

Remark 3.1 Let $\phi \in H$ be a fully irreducible element provided by Proposition 1.1. Theorem 8.5 of Dahmani, Guirardel and Osin [8] then implies that for some $m \geq 1$ the normal closure $U = ncl(\phi^m)$ of ϕ^m in $\mathrm{Out}(F_N)$ is free of infinite rank and every nontrivial element of U is fully irreducible. Since $H \leq \mathrm{Out}(F_N)$ is normal by assumption, we have $U \leq H$.

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