THE COMBINATION THEOREM AND QUASICONVEXITY

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Abstract. We show that if G is a fundamental group of a finite k-acylindrical graph of groups where every vertex group is word-hyperbolic and where every edge-monomorphism is a quasi-isometric embedding, then all the vertex groups are quasiconvex in G (the group G is word-hyperbolic by the Combination Theorem of M.Bestvina and M.Feighn). This allows one, in particular, to approximate the word metric on G by normal forms for this graph of groups.

1. Introduction

One of the basic questions in the theory of word-hyperbolic groups is to understand which amalgamated free products and HNN-extensions of hyperbolic groups are again hyperbolic.

This is a rather difficult problem which has been the subject of extensive investigation (see [Gr87], [KM98], [Pa93], [Gi96]). The most comprehensive answer was provided by the Combination Theorem of M.Bestvina and M.Feighn [BF92], [BF96]. Namely, M.Bestvina and M.Feighn give a sufficient condition which ensures that the fundamental group G of a finite graph of hyperbolic groups is word-hyperbolic. An essential part of this condition is the requirement that all the edge-monomorphisms be quasi-isometric embeddings (so that the edge groups are also word-hyperbolic and their images in the vertex groups under edge monomorphisms are quasiconvex). The following question naturally arises in this regard: assuming that the hypothesis of the Combination Theorem is satisfied, when are the vertex groups of the graph of groups quasiconvex in G?

Recall that a subgroup H of a word-hyperbolic group G is said to be *quasiconvex* in G if for some finite generating set $\mathcal{G}$ of G there is $\epsilon > 0$ such that any geodesic in the Cayley graph $\Gamma(G, \mathcal{G}$ with endpoints in H is contained in the ϵ-neighborhood of H. Quasiconvex subgroups of hyperbolic groups (which are closely related to geometrically finite Kleinian groups) are themselves word-hyperbolic.

It turns out that there are some important instances when the Combination Theorem applies but the vertex groups are not quasiconvex in the resulting group.

Example 1.1 (Mapping torus)**.** Let S be a closed oriented surface of genus at least two. Let $\phi : S \longrightarrow S$ be a pseudo-anosov homeomorphism of S. Let M be the mapping-torus manifold of ϕ, that is let M be obtained from a cylinder $S \times [0, 1]$ by gluing the top to the bottom along ϕ. Put $G = \pi_1(M)$. Then G is an HNN-extension of $H = \pi_1(S)$ along the automorphism $\phi_* : H \longrightarrow H$ induced by ϕ. That is G has the presentation

$$G = \langle H, t \,|\, t^{-1}ht = \phi_*(h), h \in H\rangle$$

The group G is word-hyperbolic since by a theorem of W.Thurston [Th77] M admits a metric of constant negative curvature. It can also be shown that the hypothesis of the Combination Theorem for this HNN-extension is satisfied, which provides an alternative explanation why G is word-hyperbolic [BF92], [Br99].

However, the base-group H of this HNN-extension is not quasiconvex in G, since H is infinite, normal and has infinite index in G (see [ABC]).

Other examples of this phenomenon can be found in [BF92], [K98], [K99].

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However, in many other situations where the Combination Theorem applies, the vertex groups are known to be quasiconvex in the resulting group. For instance, if A, B are word-hyperbolic, C is a quasiconvex subgroup of A and B which is malnormal in at least on of them, then $G = A *_C B$ is word-hyperbolic and C is quasiconvex in G [Pa93], [KM98]. (A subgroup M of G is said to be *malnormal* if for any $g \in G - M$ $g^{-1}Mg \cap M = 1$.)

Recall that an action of a group G on a simplicial tree T is called *k-acylindrical* (see [Sel97]) if no nontrivial element of G fixes point-wise a segment of length k in T. Similarly, a graph of group $\mathbb{A}$ is called *k-acylindrical* if the action of the fundamental group of the $\mathbb{A}$ (in the sense of Bass-Serre theory) on the Bass-Serre covering tree of $\mathbb{A}$ is k-acylindrical. We will also say that a group action on a tree or a graph of groups is *acylindrical* if it is k-acylindrical for some $k \geq 1$.

In this paper we show that acylindricity turns out to be responsible for the quasiconvexity of vertex groups in the context of the Combination Theorem. Namely, our main result is the following:

Theorem 1.2. *(c.f. Theorem 3.10) Let $\mathbb{A}$ be a finite acylindrical graph of groups where every vertex group is word-hyperbolic, every edge group is finitely generated and every edge-monomorphism is a quasi-isometric embedding.*

Let T be a maximal subtree in A and let $G = \pi_1(\mathbb{A}, T)$. (So that G is word-hyperbolic by the Combination Theorem.)

Then each vertex group A_v of $\mathbb{A}$ is quasiconvex in G.

Theorem 1.2 implies, in particular, that vertex groups are rational with respect to the language of geodesics on G and, moreover, with respect to any automatic language for G.

Note that the graph-of-groups decomposition of G in Example 1.1 is not acylindrical. Indeed, it is easy to see that in this case the Bass-Serre tree is an infinite line and the subgroup H fixes this line point-wise.

However, there are several important examples of acylindrical splittings which produce word-hyperbolic groups (and where Theorem 3.10 applies).

Example 1.3 (Cyclic splittings)**.** Let G be a torsion-free word-hyperbolic group and let $G = \pi_1(\mathbb{A}, *)$ be a cyclic splitting of G, that is presentation of G as the fundamental group of a graph of group where all edge groups are cyclic (e.g. JSJ-decomposition of G). Then $\mathbb{A}$ is acylindrical.

It should be noted that in the previous example it is not hard to establish quasiconvexity of the vertex groups in G without using Theorem 3.10 (see for instance, [RS97], [KM98], [K97]). The basic reason is that the edge groups are cyclic and therefore quasiconvex in G.

Example 1.4 (Separated free constructions)**.** Let A and B be word-hyperbolic groups and let C be a quasiconvex subgroup of A and B which is malnormal in A. Then the group $G = A *_C B$ is word-hyperbolic and the corresponding graph of groups is 2-acylindrical. Similarly, if K is a quasiconvex subgroup of A isomorphic to C such that any two conjugates of C and K intersect trivially, then the HNN-extension $G = \langle A, t \,|\, t^{-1}Ct = K \rangle$ is word-hyperbolic and the corresponding graph of groups is 2-acylindrical (see [K99] for a detailed argument).

Example 1.5 (3-manifold groups)**.** Let M be a closed oriented atoroidal 3-manifold. Let F be an incompressible surface in M and let M' is obtained from M by cutting open along F. Thus the fundamental group of M splits either an HNN-extension or an amalgamated free product, depending on whether M' has one or two connected components.

Suppose that $G = \pi_1(M')$ is word-hyperbolic, the boundary $\partial M'$ is not contained in the characteristic submanifold of M' and suppose that the inclusion of each component of $\partial M'$ in M' lifts to a quasiisometry between universal covers. Then the splitting of $\pi_1(M)$ corresponding to cutting M open along F is acylindrical‘[BF92]. (And so G is word-hyperbolic by the theorem of M.Bestvina and M.Feighn)

Example 1.6 (Magnus-Moldavansky splittings of one-relator groups with torsion)**.** Let

$$G = \langle x_1, \dots, x_k \,|\, R^n = 1 \rangle$$

where R is a nontrivial cyclically reduced word in $x_1, \dots, x_k$ and $n \geq 2$. Then G is known to be word-hyperbolic because of the "Spelling Theorem" of B.B.Newman [N73].

Suppose one of the generators which is involved in R occurs in R with exponent sum zero. Then (see [LS77] for details) G can be presented as an HNN-extension of another one-relator group with torsion $G = \langle H, t \,|\, t^{-1}M_1 t = M_2\rangle$, where the relator Q^n of H has shorter length than R^n and where the associated subgroups M_1 and M_2 are free (they are, in fact, Magnus subgroups of the base H and are quasiconvex in H). This HNN-extension is acylindrical, as was observed in [K96].

One of useful corollaries of Theorem 3.10 is that it allows us to approximate the word-metric (i.e. quasigeodesics) for the fundamental group of an acylindrical graph of groups by using normal forms (in the sense of Bass-Serre) for this graph of groups. Namely, we prove:

Corollary 1.7. *Suppose that $G = \pi_1(\mathbb{A}, T)$ where $\mathbb{A}$ is a finite acylindrical graph of groups and where T is a maximal tree in A. Suppose also that all vertex groups are word-hyperbolic and all edge monomorphisms are quasi-isometric embeddings. Let $X = E^+A \cup \underset{v\in VA_v}{X}$ where X_v is a finite generating set for A_v. Let p be a vertex of A. Then there exists a constant $K > 0$ with the following property.*

For any $g \in G$ there is a (K, K)-quasigeodesic with respect to d_X word W representing g of the form

$$W = V_1 \dots V_s$$

where each V_k is either $e^{\pm 1}$ for some $e \in E$ or V_k is a d_{X_v}-geodesic word for some $v \in VA$ and

$$\overline{V}_1, \dots, \overline{V}_s$$

is a reduced path from p to p with respect to the graph of groups $\mathbb{A}$. (For a formal definition of reduced paths see Section 5.)

We plan to use Theorem 1.2 and Corollary 1.7 in a future paper to obtain a criterion of quasiconvexity for an arbitrary finitely generated subgroup H of a group G satisfying the hypothesis of Theorem 1.2. Some results of this sort are obtained in [Gi97], [K97]. Such a quasiconvexity criterion can be used to build new interesting examples of coherent and locally quasiconvex (i.e. groups where all finitely generated subgroups are quasiconvex) word-hyperbolic groups out of existing ones. Recently J.McCammond and D.Wise [MW98] developed a powerful new approach which allowed them to construct many small cancellation groups which are coherent and locally quasiconvex.

The proof of Theorem 1.2 is organized as follows. First we show, using collapses of graphs of groups, that it is enough to establish Theorem 1.2 for acylindrical HNN-extensions and acylindrical amalgamated free products. We then observe that the amalgamated product case can be reduced to an HNN case via a special embedding trick. Finally we deal with the HNN-case by utilizing the method of t-strips (or t-corridors), as developed in [BRS97], [KM98], [BG96]. The proof is facilitated by the fact that for a finitely generated subgroup of a word-hyperbolic group quasiconvexity follows from the subgroup having sub-exponential (e.g. polynomial) distortion function. This allows us to use much rougher estimates. We provide a rather detailed argument for most steps in the argument (excepting for a few basic hyperbolic facts) at the expense of increasing the length of the proof.

The notion of acylindricity is close to a more algebraic condition for the edge groups of a splitting to have *finite height* in the fundamental group of a graph of groups. In [GMRS] R.Gitik, M.Mitra, E.Rips and M.Sageev showed that a quasiconvex subgroup of a hyperbolic group always has finite height. It is still not known whether the converse is true (for finitely generated subgroups). M.Mitra proved [Mi97-2] that if a hyperbolic group G splits over a hyperbolic group H as an amalgamated free product (or an HNN-extension) and if H is quasiconvex in the amalgamated subgroups (or the base of the HNN-extension) then H is quasiconvex in G if and only if H has finite height in G. In [Mi97-2] M.Mitra also pointed out that it is not clear how to generalize his argument for the case of an arbitrary graph of groups. However, it is possible to use M.Mitra's result [Mi97-2] to deduce Theorem 1.2. Namely, one can first show that for an acylindrical HNN-extension the associated subgroup has finite height and thus is quasiconvex according to Theorem 4.6 of [Mi97-2]. Then one can use the reduction of Theorem 1.2 to the HNN-case which we obtain in this paper.

Nevertheless, we decided to present a complete and independent proof of Theorem 1.2, for several reasons. First, it is advantageous to have a self-contained proof since M.Mitra, as far as we were able to determine, does not intend to publish preprint [Mi97-2]. Also, the proofs given in [Mi97-2] rely on a rather ingenious but difficult machinery developed by M.Mitra in [Mi97-2], [Mi98-1], [Mi98-2] to analyze the Cannon-Thurston map. It seems beneficial to have a direct proof that relies on more elementary considerations. Finally, we hope that the present paper will make it easier for the reader to follow the original Bestvina-Feighn article [BF92].

The author is very grateful to G.A.Swarup for drawing the author's attention to the preprint of M.Mitra [Mi97-2].

2. Some general remarks regarding quasiconvexity

Convention 2.1. If α is a path in some metric space or in a graph or if α is a word in some alphabet, we will denote by $|\alpha|$ the length of α.

Let A be a group with a finite generating set $\mathcal{A}$. For any $a \in A$ we will denote by $|a|_{\mathcal{A}}$ the word-length of a with respect to the generating set $\mathcal{A}$. We will also denote by $d_{\mathcal{A}}$ the word metric on the Cayley graph $\Gamma(A, \mathcal{A})$ induced by $\mathcal{A}$. If w is a word in $\mathcal{A}$, we denote by $\overline{w}$ the element of A represented by w.

Let α be a naturally parameterized path with origin o and terminus s. Let p, q be two points on α. We say that p is *to the right of* q on α if for any initial segment α_p of α with terminus p and any initial segment α_q of α with terminus q we have $|\alpha_q| < |\alpha_p|$.

Definition 2.2 (Quasigeodesics)**.** A path α, parameterized by arc-length, in a metric space (X, d) is called λ-quasigeodesic if for any subpath β of α with end-points p, q we have $|\beta| \le \lambda d(p, q)$.

Similarly, a path α, parameterized by arc-length, in a metric space (X, d) is called (λ, ϵ)-quasigeodesic if for any subpath β of α with end-points p, q we have $|\beta| \le \lambda d(p, q) + \epsilon$.

Let γ be a path in (X, d) with initial point o and terminal point s. Let γ' be a path in (X, d) with initial point o' and terminal point s'. We say that γ and γ' are *ϵ-Hausdorff close* if $d(o, o') \le \epsilon$, $d(s, s') \le \epsilon$ and for any point p on one of these paths there is a point q on the other path such that $d(p, q) \le \epsilon$.

We recall the important notion of a *distortion function* of a subgroup.

Definition 2.3 (Distortion function)**.** Let A be a finitely generated subgroup of a finitely generated group G. Let $\mathcal{A}$ and $\mathcal{G}$ be finite generating sets for A and G accordingly.

We say that a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is *a distortion function of* A in G if

$$|a|_{\mathcal{A}} \le f(|a|_{\mathcal{G}}) \quad \text{for any } a \in A.$$

Note that if $M = \max\{|x|_{\mathcal{G}} \, | x \in \mathcal{A}\}$ then any word of length n in $\mathcal{A}$ can be rewritten as a word of length at most Mn in $\mathcal{G}$. Therefore for any $a \in A$ we have

$$|a|_{\mathcal{G}} \le M|a|_{\mathcal{A}} \text{ that is } (1/M)|a|_{\mathcal{G}} \le |a|_{\mathcal{A}}$$

Definition 2.4. We say that a finitely generated subgroup A is *quasi-isometrically embedded in* a finitely generated group G if for some (and therefore for any) choices of generating sets there is a linear distortion function f for A in G, that is a function of the form $f(n) = Cn$, where $C \ge 1$.

Quasiconvex subgroups play an important role in the theory of word-hyperbolic groups. They are themselves finitely presented and word-hyperbolic and enjoy a number of particularly good algebraic and algorithmic properties. For example, quasiconvex subgroups are rational with respect to the language of geodesics (or in fact with respect to any automatic language) for the ambient hyperbolic group (see [GS91], [E-T92]). We refer the reader to [ABC], [KS96], [CDP90], [Gr87], [MT94], [GS91], [GMRS], [BGSS] for a detailed discussion on the subject.

We will need, however, one basic fact about quasiconvex subgroups.

Lemma 2.5. [BGSS] *Let G be a word-hyperbolic group and let H be a quasiconvex subgroup of G. Let $\mathcal{G}$ be a finite generating set for G which includes a finite generating set $\mathcal{H}$ for H.*

Then there exists a constant $\lambda > 0$ with the following property. Any $d_{\mathcal{H}}$ geodesic word w in $\mathcal{H}$ defines a λ-quasigeodesic in the Cayley graph $\Gamma(G, \mathcal{G})$ of G.

Another useful property of word hyperbolic groups states that quasi-isometrically embedded subgroups (i.e. subgroups with linear distortion) are exactly the subgroups which are quasiconvex. In fact we need a slightly stronger form of this statement which says that subgroups with sub-exponential distortion are quasiconvex.

Proposition 2.6. *Let G be a word-hyperbolic group with a finite generating set $\mathcal{G}$. Let $A \leq G$ be a finitely generated subgroup of G with a finite generating set $\mathcal{A}$. Then the following conditions are equivalent:*

1. *The subgroup A is quasiconvex in G.*
2. *The subgroup A is quasi-isometrically embedded in G.*
3. *There is a distortion function f for A in G which is* sub-exponential. *That is, for any real number $r > 1$*

$$\lim_{n\to\infty} \frac{f(n)}{r^n} = 0$$

Proof. It is well-known (see, for example, [BGSS], [GS91]) that (1) $\Leftrightarrow$ (2) and it is obvious that (2) $\Rightarrow$ (3).

The fact that (3) $\Rightarrow$ (1) is also well-known (see, for example, [Gr93], [Mi98-3]) but we are not aware of any published proof for it. For this reason we give a sketch of an argument why (3) $\Rightarrow$ (1).

Let A be a finitely generated subgroup of G. We choose a finite generating set $\mathcal{G}$ of G which includes a finite generating set $\mathcal{A}$ of A.

Similarly to the notion of a quasigeodesic, we say that a path α in $\Gamma(G, \mathcal{G})$ is an *f-distorted geodesic* if α is parameterized by arc-length and for any subpath β of α we have

$$s \leq f(d_{\mathcal{G}}(p, q))$$

where s is the length of β and p, q are the endpoints of β.

Exactly the same proof (word-by-word, using the fact that geodesics in hyperbolic spaces diverge exponentially) as that of Proposition 3.3 in [ABC] shows that for a sub-exponential function f there exists a constant D, depending on f, such that any f-distorted geodesic in $\Gamma(G, \mathcal{G})$ is D-Hausdorff close to a geodesic with the same endpoints.

Suppose now that A has sub-exponential distortion in G. Then there is a sub-exponential function f such that any $\mathcal{A}$-geodesic word W defines an f-distorted geodesic in $\Gamma(G, \mathcal{G})$ from 1 to an element $a = \overline{W} \in A$. This, by the previous remark, implies the existence of a constant D depending on f such that W is D-Hausdorff close to a $\mathcal{G}$-geodesic w from 1 to a. Therefore A is quasiconvex in G. □

We will also need the following obvious but important lemma.

Lemma 2.7. *Let $A \leq B \leq G$ be a chain of finitely generated groups. If A is quasi-isometrically embedded in G then A is also quasi-isometrically embedded in B.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$ be finite generating sets for A, B and G accordingly such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{G}$.

Since A is quasi-isometrically embedded in G, there exists a constant $C > 0$ such that for any $a \in A$ $|a|_{\mathcal{A}} \leq C|a|_{\mathcal{G}}$.

Then for any $a \in A$ we have

$$|a|_{\mathcal{A}} \geq |a|_{\mathcal{B}} \geq |a|_{\mathcal{G}} \geq (1/C)|a|_{\mathcal{A}}.$$

Therefore

$$|a|_{\mathcal{B}} \geq (1/C)|a|_{\mathcal{A}}, \quad |a|_{\mathcal{A}} \leq C|a|_{\mathcal{B}}$$

and so A is quasi-isometrically embedded in B, as required. □

3. Acylindrical splittings

We assume some familiarity of the reader with the theory of graphs of groups and of group acting on simplicial trees developed by J.Serre and H.Bass. A detailed exposition can be found in [Ser80] and [Ba93].

Definition 3.1 (Acylindricity)**.** Let $k \geq 1$ be an integer. An action of a group G on a simplicial tree T without inversions is called *k-acylindrical* if no nontrivial element of G fixes point-wise a segment of length k in T.

A splitting of a group G as the fundamental group of a graph of groups is called *k-acylindrical* if the action of G on the Bass-Serre tree corresponding to this splitting is k-acylindrical.

We will also say that an action (or a splitting) of G is *acylindrical* if it is k-acylindrical for some $k \geq 1$.

The notion of acylindricity can be re-formulated more explicitly in terms of graphs of groups. Although more technical, this approach plays an important role in applying the Combination Theorem [BF92].

When working with graphs of groups, we will adopt the following notations.

Convention 3.2. Let $\mathbb{A}$ be a graph of groups with an underlying graph A. For an edge $e \in EA$ we denote by $\alpha(e)$ the initial vertex of e and by $\omega(e)$ the terminal vertex of e. The corresponding boundary monomorphisms will be denoted by $\alpha[e]$ and $\omega[e]$ accordingly, so that:

$$\alpha[e] : A_e \longrightarrow A_{\alpha(e)} \text{ and } \omega[e] : A_e \longrightarrow A_{\omega(e)}$$

Definition 3.3 (Annulus)**.** Let $M \geq 1$ be an integer. We say that a *combinatorial M-annulus* for a graph of groups $\mathbb{A}$ is a pair $\Sigma = (p, \underline{c})$ which satisfies the following requirements.

1. p is a sequence of the form
$$p = e_1, a_1, e_2, a_2, \dots, e_{M-1}, a_{M-1}, e_M$$
where $e_i \in EA$, $e_1, \dots, e_M$ is an edge-path in A and where $a_i \in A_{\omega(e_i)}$ for $i = 1, \dots, M-1$;
2. $\underline{c}$ is a sequence of the form
$$\underline{c} = c_1, c_2, \dots, c_M$$
where $c_i \in A_{e_i}, c_i \neq 1$ for $i = 1, \dots, M$;
3. for every $i = 1, \dots, M-1$ we have
$$a_i^{-1}\omega[e_i](c_i)a_i = \alpha[e_{i+1}](c_{i+1}) \text{ in } A_{\omega(e_i)} = A_{\alpha(e_{i+1})}.$$

Such an annulus Σ is called *essential* if the sequence p is a reduced path in the graph of groups $\mathbb{A}$, that is whenever $e_{i+1} = e_i^{-1}$ we have

$$a_i \notin \omega[e_i](A_{e_i})$$

We will often think of an annulus Σ as a labeled diagram of the type shown in Figure 1.

Note that if Σ is an essential annulus as above, the path p defines a geodesic segment of length M in the Bass-Serre covering tree corresponding to the graph of groups $\mathbb{A}$. Moreover, the element $\alpha[e_1](c_1)$ fixes this segment point-wise.

Also, it is not hard to see that whenever an element $g \in G = \pi_1(\mathbb{A}, *)$ fixes a segment of length M in the Bass-Serre tree, then g is conjugate in G to $\alpha[e_1](c_1)$ for some essential M-annulus.

Thus the following statement obviously holds.

Lemma 3.4. *Let $G = \pi_1(\mathbb{A}, *)$ be the fundamental group of a graph of groups $\mathbb{A}$. Let T be the Bass-Serre tree of $\mathbb{A}$ with the associated action of G.*

Then the following are equivalent:

1. *The action of G on T is acylindrical.*
2. *There is $M \geq 1$ such that there do not exist any essential M-annuli for $\mathbb{A}$.*

The definition below is one of the standard notions in Bass-Serre theory.

Definition 3.5 (Refinement, collapse)**.** Let $\mathbb{A}$ be a graph of groups and let $v \in VA$. Suppose that $A_v = \pi_1(\mathbb{B}, T')$ where $\mathbb{B}$ is a graph of groups and T' is a maximal subtree of B.

We will say that the splitting $A_v = \pi_1(\mathbb{B}, T')$ is a splitting *relative* to $\mathbb{A}$ if for every edge e of EA with $\alpha_A(e) = v$ there exists a vertex $v(e) \in VB$ such that $\alpha[e](A_e) \leq B_{v(e)}$. In this case we define a new graph of groups $\mathbb{D}$, called the *refinement* of $\mathbb{A}$ at the vertex v with respect to $\mathbb{B}$ as follows. The vertex set VD of $\mathbb{D}$ is $(VA - \{v\}) \cup VB$ and the edge set ED of D is $EA \cup EB$. The origin and terminus maps are

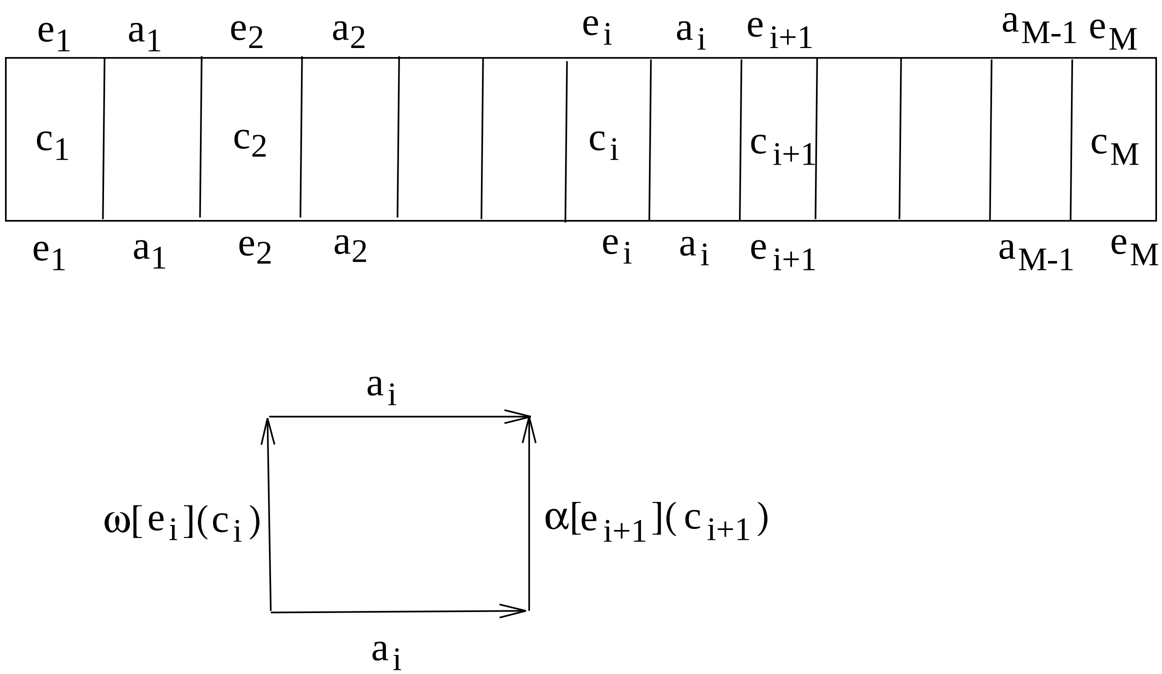

FIGURE 1. Annulus

inherited from A and B except that for $e \in EA$ with $\alpha_A(e) = v$ we put $\alpha_D(e) = v(e)$. The vertex and edge groups as well as the boundary monomorphisms for $\mathbb{D}$ are inherited from $\mathbb{A}$ and $\mathbb{B}$ in the obvious way.

Notice that in this situation T together with T' give rise to the maximal subtree T'' of D and there is a canonical isomorphism $\pi_1(\mathbb{D}, T'') \cong \pi_1(\mathbb{A}, T)$.

The operation inverse to a refinement is called a *collapse*.

Proposition 3.6 (Collapses of acylindrical splittings are acylindrical)**.** *Let $\mathbb{A}$ be a finite graph of groups with finitely generated vertex and edge groups.*

Let $X_1, \dots, X_s$ be connected disjoint subgraphs of A. Let $\mathbb{B}$ be the graph of groups obtained from $\mathbb{A}$ by collapsing each of $X_1, \dots, X_s$ into a vertex.

Then the following holds:

1. *For each X_i the induced graph of groups $\mathbb{X}_i$ is acylindrical.*
2. *If $\mathbb{A}$ is acylindrical then $\mathbb{B}$ is acylindrical.*

Proof. Choose maximal subtrees $Y_1, \dots, Y_s$ in $X_1, \dots, X_s$ accordingly. Choose a maximal subtree Y of A so that Y contains each of Y_i. Put $G = \pi_1(\mathbb{A}, Y)$ and let T be the universal covering Bass-Serre tree of $\mathbb{A}$ with the standard action of G. Thus $T//G = A$. If $e \in EA$ is an edge of A, we say that an edge $f \in ET$ is *of type* e if the image of f under the natural quotient map $p : T \longrightarrow A$ is equal to e.

For each X_i put $G_i = \pi_1(\mathbb{X}_i, Y_i)$. It is clear that for each X_i the minimal G_i-invariant subtree T_i in T is exactly the universal covering Bass-Serre tree of $\mathbb{X}_i$. Since the action of G on T is k-acylindrical, the action of any subgroup of G on T is also k-acylindrical. Hence the action of G_i on T_i is k-acylindrical and (1) is verified.

In the tree T consider all maximal subtrees Q such that Q does not contain any e-type vertices for $e \in Y_1 \cup \cdots \cup Y_s$. Let T' be the graph obtained from T by collapsing each such Q into a vertex. It is clear that the action of G on T factors through to an action of G on T' and that T' is a tree. It is also clear that $T'//G = B$. Moreover, it follows from the explicit description of T in terms of $\mathbb{A}$ that the graph of groups structure inherited by $T'//G = B$ from the action of G on T' is precisely $\mathbb{B}$.

Suppose now that $\mathbb{A}$ is k-acylindrical for some integer $k \geq 1$. This means that the action of G on T is k-acylindrical. We claim that the action of G on T' is also k-acylindrical. Indeed, suppose that $g \in G$, $g \neq 1$ fixes point-wise an edge-path without backtracks $f_1, \dots, f_k$ in T'. By definition of T' this means that g fixes a segment in T of the form $f_1, u_1, f_2, \dots, u_{k-1}, f_k$ where each u_i is a segment (possibly trivial) which only contains type-e edges for $e \in Y_1 \cup \dots \cup Y_s$. Thus g fixes a segment of length at least k in T which yields a contradiction.

This completes the proof of Proposition 3.6. □

the following statement is a particular case of the Combination Theorem of M.Bestvina and M.Feighn [BF92], [BF96].

Theorem 3.7 (Combination Theorem)**.** *Let $\mathbb{A}$ be a finite graph of groups such that*

1. *all vertex groups of $\mathcal{A}$ are word-hyperbolic;*
2. *every edge group is finitely generated and every edge monomorphism is a quasi-isometric embedding;*
3. *the graph of groups $\mathbb{A}$ is acylindrical.*

Then $G = \pi_1(\mathbb{A}, *)$ is word-hyperbolic.

The reader is referred to [K98] and [K99] for a more detailed discussion on the statement and the applications of the Combination Theorem.

We will establish the following theorem in the next section (see Proposition 4.15.

Theorem 3.8. *Let $G = \langle A, t \,|\, t^{-1}Ut = V\rangle$ be an acylindrical HNN-extension of a word-hyperbolic group A such that subgroups U and V are quasiconvex in A (so that G is word-hyperbolic by Theorem 3.7). Then A is quasiconvex in G.*

Theorem 3.9. *Let $G = A *_C B$ be an acylindrical splitting where A, B are word-hyperbolic and C is quasiconvex in both A and B. (Thus G is word-hyperbolic itself.)*

Then A, B and C are quasiconvex in G.

Proof. Let A and B be groups such that $U \leq A$, $V \leq B$ are their isomorphic subgroups and let $\phi : U \longrightarrow V$ be an isomorphism. Suppose further that A, B are word-hyperbolic and that U, V are quasiconvex in A, B accordingly. Put

$$G = A *_{U=_\phi V} B \text{ or just } G = A *_{U=V} B \tag{1}$$

to be the corresponding amalgamated product.

Then G can be embedded as a subgroup of the following HNN-extension:

$$G' = \langle (A * B), t \,|\, t^{-1}ut = \phi(u), u \in U\rangle = \langle (A * B), t \,|\, t^{-1}Ut = V\rangle \tag{2}$$

Namely, the subgroup $H = \langle A, t^{-1}Bt\rangle$ of G' is easily seen to be isomorphic to G.

Claim. If the splitting (1) is acylindrical then the splitting (2) is also acylindrical.

Indeed, consider the graphs of groups $\mathbb{X}$ and $\mathbb{Y}$ corresponding to (1) and (2) as shown in Figure 2. We will denote the edge of X by e and the edge of Y by t.

Suppose $\Sigma = (p, \underline{c})$ is an essential M-annulus for $\mathbb{Y}$. Note that $g^{-1}Ug \cap V = 1$ for any $g \in A * B$. Therefore p does not contain subsequences of the form t, g, t or t, g, t^{-1} where $g \in A * B$. Moreover, if $g^{-1}ug \in U, u \in U, u \neq 1$ for some $g \in A * B$ then $g \in A$. Similarly, if $g^{-1}vg \in V, v \in V, v \neq 1$ for some $g \in A * B$ then $g \in B$. Thus if t, g, t^{-1} is a subsequence of p, then $g \in B$ and if t^{-1}, g, t is a subsequence of p then $g \in A$.

Therefore if we formally replace each t in p by e and each t^{-1} by e^{-1}, we will get an M-annulus Σ_1 for $\mathbb{X}$.

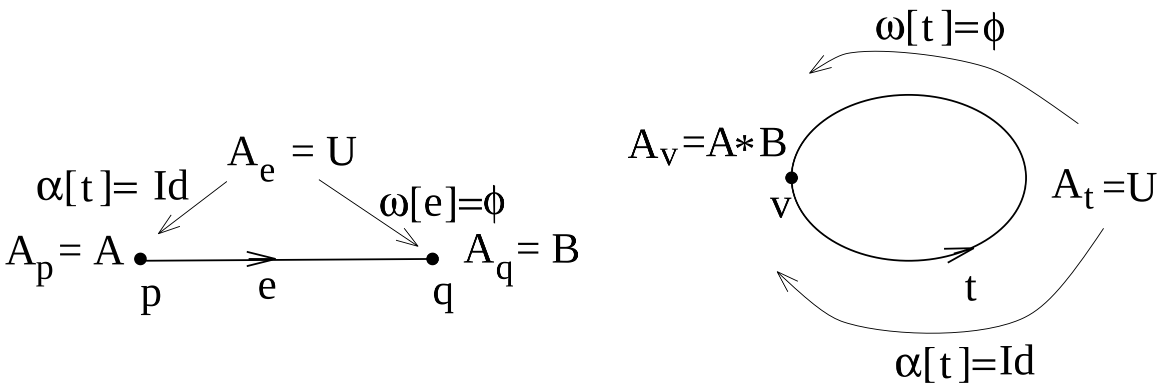

FIGURE 2. Graphs of groups $\mathbb{X}$ and $\mathbb{Y}$

Since Σ is essential for $\mathbb{Y}$, whenever t, b, t^{-1} is a subsequence of p, we have $b \in B - V$. Similarly, whenever t^{-1}, a, t is a subsequence of p, we have $a \in A - U$. Thus the annulus Σ_1 is also essential. We showed that the existence of an essential M-annulus for Y implies the existence of an essential M-annulus for X. This proves the Claim.

Thus the Claim holds and HNN-splitting (2) is acylindrical. Note also that the group $A * B$ is word-hyperbolic and the groups A, B are quasi-isometrically embedded in $A * B$. Since U, V are quasi-isometrically embedded in A and B accordingly, this implies that U and V are quasi-isometrically embedded in $A * B$. Thus all the conditions of Theorem 3.8 are satisfied and therefore $A * B$ is quasiconvex in G'. Since A, B are quasiconvex in $A * B$, this implies that A and B (and hence $t^{-1}Bt$) are quasiconvex in G'. By Lemma 2.7 this means that $A, t^{-1}Bt$ are quasiconvex in $G = \langle A, t^{-1}Bt \rangle$ which completes the proof of Theorem 3.9.

□

Theorem 3.10. *(c.f. Theorem 1.2) Let $\mathbb{A}$ be a finite acylindrical graph of groups where every vertex group is word-hyperbolic, every edge group is finitely generated and every edge-monomorphism is a quasi-isometric embedding.*

Let Y be a maximal subtree in A and let $G = \pi_1(\mathbb{A}, Y)$. (So that G is word-hyperbolic by the Combination Theorem.)

Then each vertex group A_v of $\mathbb{A}$ is quasiconvex in G.

Proof. We will prove this theorem by induction on the number of edges in A.

If A has no edges, then A is a single vertex and there is nothing to prove. Suppose now that A has $n > 1$ edges and that Theorem 3.10 has been proved for all graphs with fewer than n edges.

Choose an edge e of A. If $A - e$ is connected, put $X_1 = A - e$. If $A - e$ is disconnected, put X_1 to be the connected component of $A - e$ containing the initial vertex of e and put X_2 to be the connected component of $A - e$ containing the terminal vertex of e. Choose a maximal tree Y in A so that Y contains maximal trees Y_1 and Y_2 of X_1 and X_2 (or just of X_1 if $A - e$ is connected).

Let $\mathbb{B}$ be the graph of groups obtained from $\mathbb{A}$ by collapsing X_1 and X_2 (or just of X_1 if $A - e$ is connected). Thus B is either a single edge with two distinct endpoints or a single edge with equal endpoints.

Note that every $\mathbb{X}_i$ acylindrical by Proposition 3.6 and has fewer than n edges. Thus by induction the vertex groups of $\mathbb{X}_i$ are quasiconvex in $\pi_1(\mathbb{X}_i, Y_i)$. We also know that the images of A_e under the boundary monomorphisms are quasiconvex in the appropriate vertex groups of $\mathbb{A}$. Therefore the images of A_e under the boundary monomorphisms are quasiconvex in the vertex groups $\pi_1(\mathbb{X}_i, Y_i)$ of $\mathbb{B}$.

Thus by Theorem 3.9 and Theorem 3.8 the groups $\pi_1(\mathbb{X}_i, Y_i)$ are quasiconvex in G. Since we already know by induction that vertex groups of $\mathbb{X}_i$ are quasiconvex in $\pi_1(\mathbb{X}_i, Y_i)$ this implies that every vertex group of $\mathbb{A}$ is quasiconvex in G.

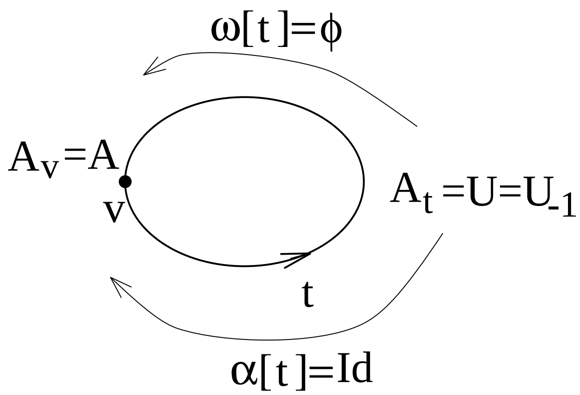

FIGURE 3. HNN-extension graph of groups $\mathbb{E}$

Thus Theorem 3.10 is proved. □

4. THE HNN-EXTENSION CASE

It remains to prove Theorem 3.8 which deals with an acylindrical HNN-extension.

Let A be a word-hyperbolic group and let U_{-1}, U_1 be isomorphic quasiconvex subgroups of A. Let $\phi : U_{-1} \longrightarrow U_1$ be an isomorphism. Choose a finite generating set $\mathcal{U}_{-1}$ of U_{-1}. Then $\mathcal{U}_1 = \phi(\mathcal{U}_{-1})$ is a finite generating set of U_1. Choose a finite generating set $\mathcal{A}$ of A so that $\mathcal{U}_{-1} \subseteq \mathcal{A}$ and $\mathcal{U}_1 \subseteq \mathcal{A}$. Let $A = \langle \mathcal{A} \,|\, R \rangle$ be a finite presentation of A.

Consider the HNN-extension presentation

$$G = \langle \mathcal{A}, t \,|\, t^{-1}xt = \phi(x) \text{ for each } x \in \mathcal{U}_{-1}, R \rangle \tag{3}$$

We denote $X = \mathcal{A} \cup t$. Thus

$$G = \langle A, t \,| t^{-1} U_{-1} t = U_1 \rangle$$

We will fix presentation (3) till the end of this section.

Put $U = U_{-1}$. We can think of G as the fundamental group of a graph of groups $\mathbb{E}$ shown in Figure 3. We will denote the edge of $\mathbb{E}$ by t and the vertex of $\mathbb{E}$ by v. The vertex group is $A_v = A$ and the edge group is $A_e = U = U_{-1}$. The boundary monomorphisms are defined as $\alpha[t] = Id_U : U_{-1} \longrightarrow U_{-1}$ and $\omega[t] = \phi : U_{-1} \longrightarrow U_1$.

Let D be a Van-Kampen diagram with respect to presentation (3) (see [LS77] for an overview of Van-Kampen diagrams over group presentations). Note that each relator of (3) involving t, involves t exactly twice.

For this reason in any Van-Kampen diagram with respect to presentation (3) all t-relators are arranged in t-rings or t-strips starting and ending at the boundary of the diagram, as shown in Figure 4.

[To be more formal, we say that two t-cells in D are elementary t-equivalent if they share an edge labeled by t. Elementary t-equivalence generates an equivalence relation on the set of t-cells in D. The union of all cells in an equivalence class is called a *t-chain*. The interior of a t-chain is either homotopically equivalent to an interval, in which case the chain is called a *t-strip*, or it is homotopically equivalent to a circle, in which case the chain is called a *t-ring*. In Figure 4 the edges transversal to the boundaries of t-chains are labeled by t.] Note that the boundaries of t-strips and t-rings may intersect, although their interiors are disjoint.

Every t-strip has two boundary components with a $\mathcal{U}_{-1}$-word and a $\mathcal{U}_1$-word written on them. We will refer to these paths in a Van-Kampen diagram as *U-paths* or *U-arcs*.

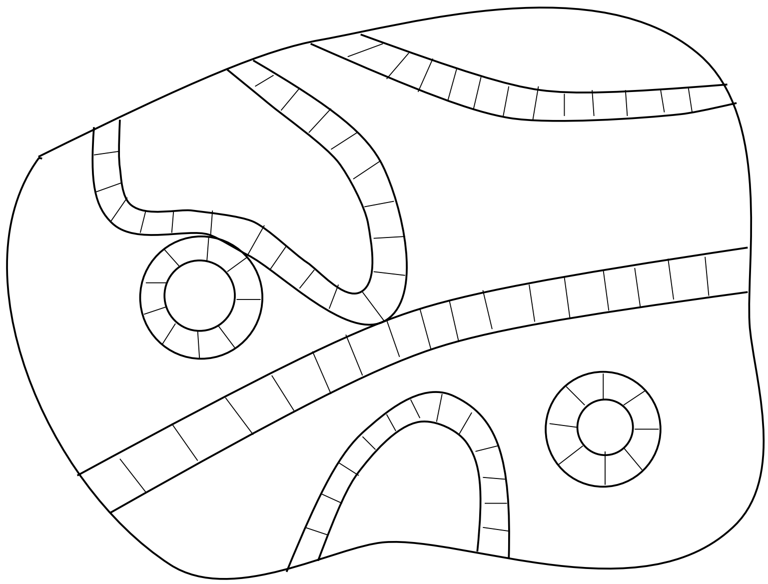

FIGURE 4. All t-relators are arranged in t-strips and t-rings.

The method of using t-strips to study Van-Kampen diagrams over HNN-extensions and other group presentations is well-established and has been used in a number of papers. A more detailed exposition of this approach can be found in [KM98], [BRS97], [BG96] and other sources.

Definition 4.1. Let w be a word in X representing 1 in G. A Van-Kampen diagram D over (3) with the boundary label w is called *t-minimal* if any other Van-Kampen diagram with the boundary label w involves no more t-relators than does D.

The following lemma is an easy corollary of the definitions.

Lemma 4.2. *A t-minimal diagram over* (3) *does not contain any t-rings.*

Proof. Suppose D is a t-minimal diagram with boundary label w and suppose that D contains a t-ring.

Consider the innermost t-ring in D. Let U be the label of the inner boundary of this t-ring and let V be the label of its outer boundary. Since the ring is innermost, its inner boundary bounds in D a Van-Kampen diagram over A with boundary label U. Thus U represents the identity element of A. By definition of the HNN-extension G this implies that V also represents the identity in A. Therefore there exists a Van-Kampen diagram Q over A (which therefore has no t-cells) with boundary label V.

Let D' be obtained from D by cutting out the subdiagram bounded by V (including the t-ring) and pasting in Q. Clearly D' has fewer t-cells than D and the boundary label of D' is still w. This contradicts our assumption that D is t-minimal. □

Convention 4.3. If α is a path in a Van-Kampen diagram or a Cayley graph, we denote by $|\alpha|$ the length of α. If p, q are points in a Van-Kampen diagram D over (1) which are joined in D by a path labeled by an $\mathcal{A}$-word v, we denote by $\bar{d}(p,q)$ the length $|\overline{v}|_{\mathcal{A}}$ (it is easy to see that this definition does not depend on the choice of v).

Recall also that a naturally parameterized path α in a metric space is called *λ-quasigeodesic* if for any subpath β of α with endpoints p, q we have $|\beta| \le \lambda d(p,q)$.

Lemma 4.4. *Let D be a t-minimal Van-Kampen diagram with respect to* (3)*. Let α be a $\mathcal{U}_i$-word which is the label of a U_i-arc in D (where $i = \pm 1$). Then α is a $\mathcal{U}_i$-geodesic (and thus the U_i arc in D does not have any self-intersections).*

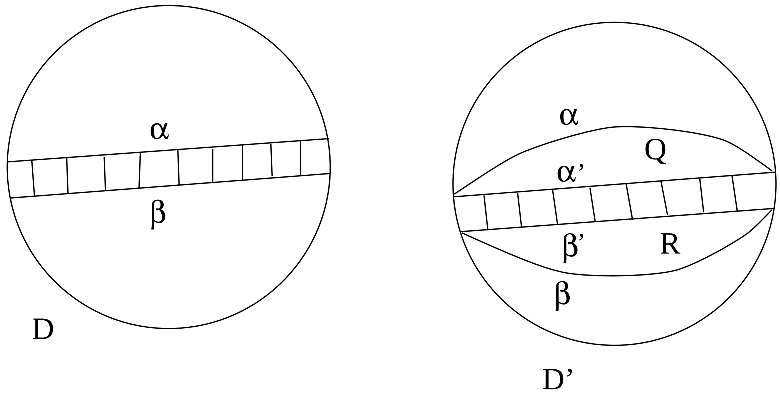

FIGURE 5

Proof. Let α be a $\mathcal{U}_i$-word which is the label of the U_i-boundary of a t-strip S in D. Let β be the U_{-i}-word written on the U_{-i}-boundary of S. Thus $\phi^i(\overline{\alpha}) = \overline{\beta}$ and $|\alpha| = |\beta|$. Note that $|\alpha|$ is equal to the number of t-relators in S.

Suppose that α is not $\mathcal{U}_i$-geodesic and there is a $\mathcal{U}_i$-word α' such that $|\alpha'| < |\alpha|$ and $\overline{\alpha} = \overline{\alpha'}$ in U_i.

Put β' to be the U_{-i}-word obtained from α by applying ϕ^i letter-wise. Thus

$$\overline{\beta'} = \overline{\beta} = \phi^i(\overline{\alpha}) = \phi^i(\overline{\alpha'})$$

In particular we can make a t-strip S' with the word α' written on its U_i-boundary and the word β' written on its U_{-i}-boundary. The t-strip S' involves $|\alpha'| = |\beta'|$ t-relators.

Let Q and R be A-Van-Kampen diagrams with the boundary labels $\alpha'\alpha^{-1}$ and $\beta(\beta')^{-1}$ accordingly (so that Q and R do not involve any t-relators).

We can now form a Van-Kampen diagram D' from D by cutting out the t-strip S and pasting in the t-strip S' with Q and R attached to it, as shown in Figure 5. Notice that the boundary labels of D and D' are the same. On the other hand D' involves $|\alpha| - |\alpha'|$ fewer t-relators than does D. This contradicts t-minimality of D. □

We will assume that the reader is familiar with the basics of the theory of word-hyperbolic groups. A detailed background information can be found in [Gr87], [ABC], [CDP90], [GH90].

Let $\delta > 1$ be such that all geodesic triangles in the Cayley graph $\Gamma(A, \mathcal{A})$ are δ-thin. We will need the following three lemmas which follow from the elementary properties of word-hyperbolic groups. We therefore leave the proofs to the reader.

Lemma 4.5 (Thin polygons)**.** *For any $\lambda > 0$ there exists a constant $C_0 = C_0(\delta, \lambda) > 1$ such that the following holds.*

Suppose we have a quasigeodesic polygon in $\Gamma(A, \mathcal{A})$ with λ-quasigeodesic sides $w, \alpha_1, \dots, \alpha_n$ as shown in Figure 6.

Then w can be broken into $m \le n$ subsegments $w = w_1 \cup w_2 \cup \dots \cup w_m$ such that each w_i is nC_0-Hausdorff close to a subsegment of one of α_j.

The above statement can be easily proved by induction on n using the fact that geodesic triangles in Gromov-hyperbolic spaces are thin. It also follows immediately from the properties of approximating trees (see [Gr87], [Bo98]).

Convention 4.6 (Constants)**.** Since U_i are quasiconvex in A, there exists a constant $\lambda > 2$ such that any $\mathcal{U}_i$-geodesic word α is λ-quasigeodesic with respect to $d_{\mathcal{A}}$. Let $C_0 = C_0(\lambda, \delta)$ be the constant whose existence follows from Lemma 4.5. We fix these constants C_0 and λ till the end of this section.

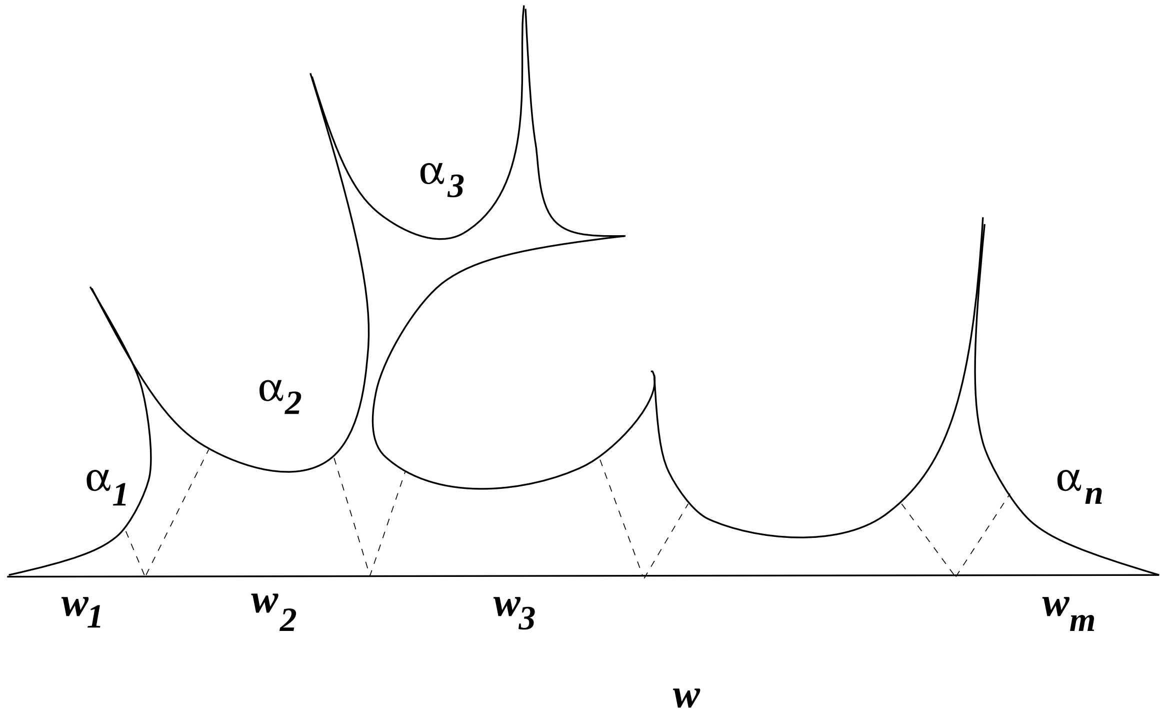

FIGURE 6. Thin polygons

Lemma 4.7. *There exist constants $C_1 > C_0 > 1$ and $Q > 4\lambda C_1 > 1$ such that the following holds.*

Suppose that α and β are λ-quasigeodesic paths in $\Gamma(A, \mathcal{A})$ from a_1 to a_2 and from b_1 to b_2 respectively. Suppose further that $d_{\mathcal{A}}(a_1, b_1) \le nC_0$ and $d_{\mathcal{A}}(a_2, b_2) \le nC_0$ where $n \ge 0$ is an integer. Then

1. *For any point p on α there is a vertex q on β such that $d_{\mathcal{A}}(p, q) \le nC_1$.*
2. *If p and p' are points on α and p' is to the right of p and $d_{\mathcal{A}}(p, p') \ge nC_2$ then there are points q, q' on β such that q' is to the right of q and $d_{\mathcal{A}}(p, p'), d_{\mathcal{A}}(q, q') \le nC_1$.*

Lemma 4.8. *There exists a constant $H > 1$ with the following property.*

Suppose we have a quadrilateral Δ in $\Gamma(A, \mathcal{A})$ whose corners are vertices a_1, a_2, b_1, b_2 of $\Gamma(A, \mathcal{A})$ and whose sides are paths x, y, α, β where x, y are $\mathcal{A}$-geodesics and α, β are λ-quasigeodesics with respect to $d_{\mathcal{A}}$ (see Figure 7). Put $M = |x| + |y|$.

Then

1. *For any point p on α such that $\min\{d_{\mathcal{A}}(p, a_1), d_{\mathcal{A}}(p, a_2)\} \ge M + H$ there is a vertex p' on β such that $d_{\mathcal{A}}(p, p') \le H$.*
2. *Suppose p, q are points on α such that q is to the right of p on α. Suppose further that $\min\{d_{\mathcal{A}}(p, a_1), d_{\mathcal{A}}(p, a_2)\} \ge M + H$ and $\min\{d_{\mathcal{A}}(q, a_1), d_{\mathcal{A}}(q, a_2)\} \ge M + H$ and that $d_{\mathcal{A}}(p, q) \ge H$.*

   Then there are vertices p', q' on β such that $d_{\mathcal{A}}(p, p') \le H$, $d_{\mathcal{A}}(q, q') \le H$ and q' is to the right of p' on β.

The following lemma will play an important part in our argument, as it will enable us to show that certain annuli are essential.

Lemma 4.9. *Suppose D is a t-minimal Van-Kampen diagram over (3) as shown in Figure 8, where*

1. *The words α and α_1 are geodesics in $\mathcal{U}_{-1}$.*
2. *The word w and is an $\mathcal{A}$-word with $|\overline{w}|_A \le H$.*
3. *The subdiagram D_0 of D with boundary label $w\alpha w^{-1}{\alpha_1}^{-1}$ does not involve any t-relators.*
4. *$|\alpha| > 3\lambda H$.*

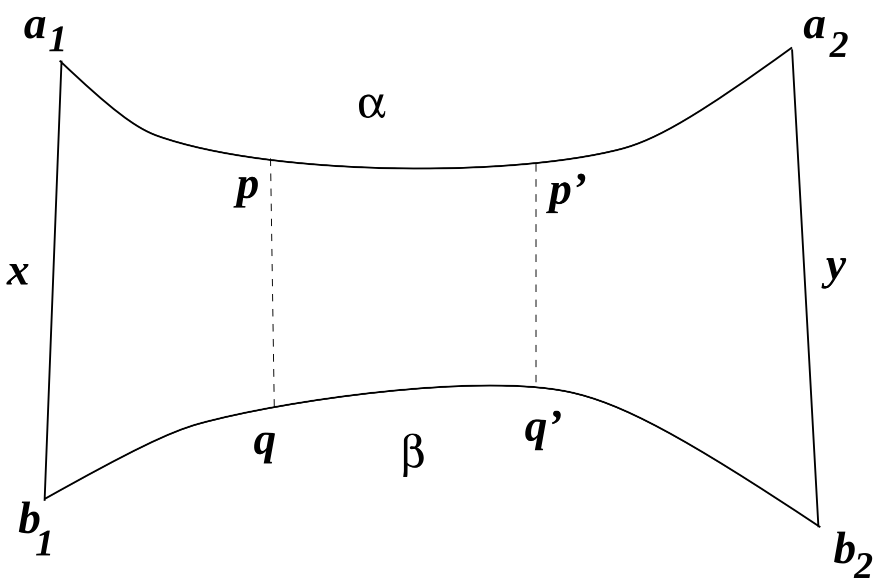

FIGURE 7. Thin quasigeodesic quadrilateral

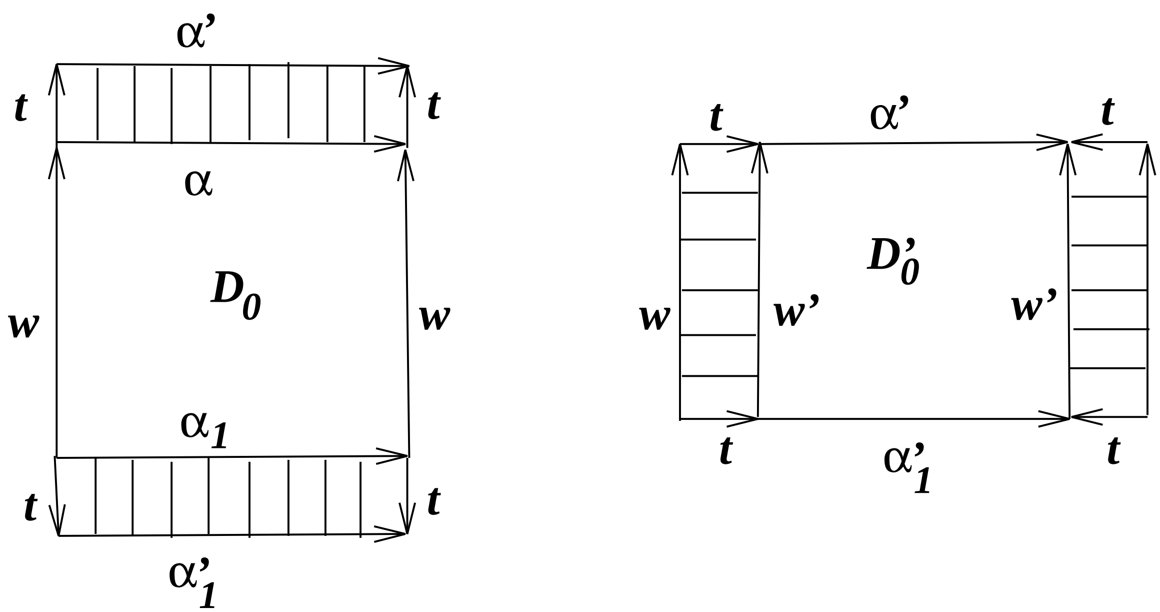

FIGURE 8. Thin quasigeodesic quadrilateral

Then $\overline{w} \notin U_{-1}$.

Proof. Suppose D is as in Lemma 4.9 and $|\alpha| > 3\lambda H$. Suppose that Lemma 4.9 does not hold and $\overline{w} \in U_{-1}$.

It is clear that if we glue two A-diagrams to D on the left and on the right along the paths labeled w, the resulting diagram will have the same number of t-relators as D and will still be t-minimal. Therefore, after modifying D if necessary, we may assume that w is a $\mathcal{U}_{-1}$-geodesic word and so $|w| \leq \lambda H$.

Note that $\overline{w\alpha w^{-1}} = \overline{\alpha}_1$ in U_{-1}. Therefore

$$\phi(\overline{w})\phi(\overline{\alpha})\phi(\overline{w})^{-1} = \phi(\overline{\alpha}_1), \text{ that is } \phi(\overline{w})\overline{\alpha'}\phi(\overline{w})^{-1} = \overline{\alpha'}_1$$

Let w' be the $\mathcal{U}_1$-word obtained from w by applying ϕ letter-wise, so that w' represents $\phi(\overline{w})$. Then, obviously, w' is a $\mathcal{U}_1$-geodesic and $|w'| = |w|$. Since $\overline{w'\alpha'(w')^{-1}\alpha_1^{-1}} = 1$ in A, there exists an A-diagram D_0' with the boundary label $w'\alpha'(w')^{-1}(\alpha_1')^{-1}$. Let D' be the diagram, shown in Figure 8, which is obtained by attaching to D_0' along the paths labeled w' the t-strips corresponding to the relation $\overline{t^{-1}wt} = \overline{w'}$.

Note that the boundary label of D_0' is the same as the boundary label of D. The t-area of D_0' is equal to $2|w|$. Recall that by assumption of Lemma 4.9 $|w| \leq \lambda H$ and $|\alpha| > 3\lambda H$. The t-area of D is equal to $|\alpha| + |\alpha_1|$. Also by the triangle inequality with respect to $d_{\mathcal{U}_{-1}}$ $|\alpha_1| \geq |\alpha| - 2|w|$. Recall that $\lambda > 1$.

Hence

$$|\alpha| + |\alpha_1| \geq |\alpha| + |\alpha| - 2|w| \geq 2|\alpha| - 2\lambda H >$$
$$6\lambda H - 2\lambda H = 4\lambda H \geq 2|w|$$

Thus $|\alpha| + |\alpha_1| > 2|w|$ which contradicts t-minimality of D. □

The following lemma is a crucial step in our argument; it demonstrates the emergence of an essential annulus.

Lemma 4.10. *There exists a constant $E \geq 1$ such that the following holds.*

Suppose $n \geq 1$, $m \geq 1$ are integers and D is a t-minimal diagram as shown in Figure 9 (upper diagram), where

1. *Each of x_i and y_i is a $\mathcal{A}$-word with $|\overline{x_i}|_{\mathcal{A}} \leq C_1 n$, $|\overline{y_i}|_{\mathcal{A}} \leq C_1 n$.*
2. *For each i we have $t_i = t^{\pm 1}$ and S_i is a t-strip.*
3. *Each of the subdiagrams in D bounded by $x_i, y_i, \gamma_i', \gamma_{i+1}$ is a diagram over A and does not involve any t-relators.*
4. *Each word γ_i (where $1 \leq i \leq m$) written on the lower boundary of the i-th t-strip (so that γ is a $\mathcal{U}_{\pm 1}$-geodesic word).*
5. *$|\gamma_1'| \geq nE^m$.*

Then there exists an essential m-annulus with respect to HNN-extension (3).

Proof. Note that γ_i and γ_i' are freely reduced words and the paths corresponding to them in D do not have any self-intersections (since D is t-minimal). We will often identify γ_i and γ_i' with the paths corresponding to them in D.

In the proof of this lemma if p, q are points on γ_i in D, we denote by $|p, q|$ the length of the γ_i-segment between p and q. Similarly, if p, q are points on γ_i' in D, we denote by $|p, q|$ the length of the γ_i'-segment between p and q. For each $i = 1, \ldots, m$ let o_i and s_i be the initial vertex and the terminal vertex accordingly of the path labeled by γ_i in D. Similarly, for $i = 1, \ldots, m$ let o_i' and s_i' be the initial vertex and the terminal vertex accordingly of the path labeled by γ_i'.

Let N_0 be the number of elements in A of $\mathcal{A}$-length at most H. Thus for any integer $k \geq 1$ the number of ordered k-tuples $(e_1, \ldots, e_k)$ where $e_i \in A$ and $|e_i|_{\mathcal{A}} \leq H$ is N_0^k. Put $N = N_0^{m-1}$. Put $B = 1000\lambda^2 C_1 H^2$.

Suppose that $|\gamma_1| = |\gamma_1'| \geq 3\lambda B^m (N+1) n$.

Claim. For $i = 1, \ldots, m$ there exist vertices $a_{1,i}', \ldots a_{N+1,i}'$ on γ_i' and $a_{1,i}, \ldots, a_{N+1,i}$ on γ_i such that

1. the point $a_{j,i}$ corresponds to $a_{j,i}'$ in the t-strip S_i (that is they are joined by a t-edge);
2. for each $i = 1, \ldots, m$ we have $\bar{d}(a_{j,i}', o_i') \geq nB^{m-i+1}$ and $\bar{d}(a_{j,i}', s_i') \geq nB^{m-i+1}$
3. for each i if $j < j_1$ then $a_{j,i}'$ appears to the left from $a_{j_1,i}'$ on the path γ_i' and $\bar{d}(a_{j,i}', a_{j+1,i}') \geq B^{m-i+1}$
4. for each i, j we have $\bar{d}(a_{j,i}', a_{j,i+1}) \leq H$.

We will construct these points inductively.

Step 0. First, let $a_{1,1}'$ be the point on γ_1' such that the length of the initial segment of γ_1' from o_1' till $a_{1,1}'$ is $\lambda B^m n$.

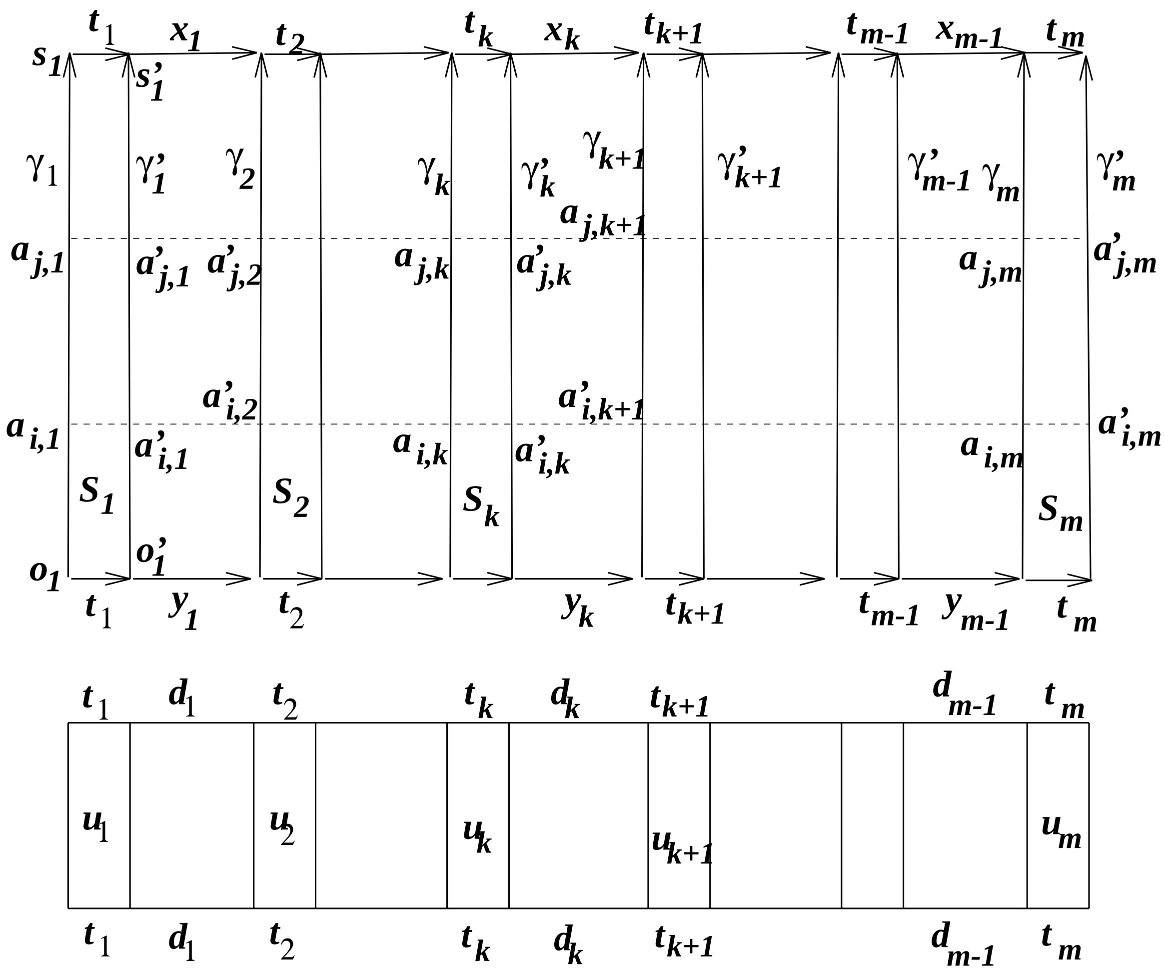

FIGURE 9. The emergence of an annulus

Since $|\gamma'_1| \geq 3\lambda B^m(N+1)n$, we can find vertices $a'_{2,1}, \dots, a'_{N+1,1}$ on γ'_1 such that $|a'_{j,1}, a'_{j_1,1}| = \lambda B^m |j - j_1|$ and that $a'_{j_1,1}$ is to the right of $a'_{j,1}$ on γ'_1 if $j < j_1$. Note that since $|\gamma'_1| \geq 3\lambda B^m(N+1)n$ for $a'_{1,1}$ and $a'_{1,N+1}$ the distances along γ'_1 to both endpoints of γ'_1 are at least $\lambda n B^m$. The same applies to each of $a'_{j,1}$. Thus for each j we have

$$\bar{d}(a'_{j,1}, o'_1) \geq nB^m, \quad \bar{d}(a'_{j,1}, s'_1) \geq nB^m.$$

Moreover, since γ'_1 is a λ-quasigeodesic with respect to $d_{\mathcal{A}}$ and $|a'_{j,1}, a'_{j+1,1}| = \lambda B^m$ we have $\bar{d}(a'_{j,1}, a'_{j+1,1}) \geq B^m$.

We set $a_{j,1}$ to be the vertices on γ_1 corresponding to $a'_{j,1}$.

Inductive step.

Suppose $a_{j,i}$ and $a'_{j,i}$ are already constructed for $j = 1, \dots, k$ where $k < m$. We will now construct $a_{j,k+1}$ and $a'_{j,k+1}$.

By Lemma 4.4 γ_{k+1} and γ'_k are $\mathcal{U}_{\pm 1}$-geodesics since D is t-minimal. By the inductive hypothesis $\bar{d}(a'_{j,k}, o'_k), \bar{d}(a'_{j,k}, s'_k) \geq nB^{m-k+1} \geq 2nC_1 + H$. Therefore by Lemma 4.8 there exist vertices $a_{1,k+1}, \dots, a_{N+1,k+1}$ on the path γ_{k+1} such that $\bar{d}(a'_{j,k}, a_{j,k+1}) \leq H$.

Let $a'_{1,k+1}, \dots, a'_{N+1,k+1}$ be the vertices on the path γ_{k+1} corresponding (in the t-strip S_{k+1}) to the points $a_{1,k+1}, \dots, a_{N+1,k+1}$. By inductive hypothesis $\bar{d}(a'_{j,k+1}, o'_{k+1}), \bar{d}(a'_{j,k+1}, s'_{k+1}) \geq nB^{m-k+1}$.

By the triangle inequality

$$\bar{d}(a_{j,k+1}, o_{k+1}) \geq \bar{d}(a'_{j,k}, o'_k) - H - nC_1 \geq nB^{m-k+1} - 2nH \geq nB^{m-k+1}/\lambda$$

because of the choice of B and since $H > C_1$. Similarly,

$$\bar{d}(a_{j,k+1}, s_{k+1}) \geq \bar{d}(a'_{j,k}, s'_k) - H - nC_1 \geq nB^{m-k+1} - 2nH \geq nB^{m-k+1}/\lambda.$$

Therefore $|a_{j,k+1}, o_{k+1}| = |a'_{j,k+1}, o'_{k+1}| \geq nB^{m-k+1}/\lambda$ and $|a_{j,k+1}, s_{k+1}| = |a'_{j,k+1}, s'_{k+1}| \geq nB^{m-k+1}/\lambda$. Since γ'_{k+1} is λ-quasigeodesic with respect to $d_{\mathcal{A}}$, this implies that $\bar{d}(a'_{j,k+1}, o'_{k+1}) \geq nB^{m-k+1}/\lambda^2 \geq nB^{m-k}$ and $\bar{d}(a'_{j,k+1}, s'_{k+1}) \geq nB^{m-k+1}/\lambda^2 \geq nB^{m-k}$ (we are using the choice of B to deduce the last two inequalities).

Moreover by the triangle inequality

$$\bar{d}(a_{j,k+1}, a_{j+1,k+1}) \geq \bar{d}(a_{j,k}, a_{j+1,k}) - 2H \geq B^{m-k+1} - 2H \geq B^{m-k+1}/\lambda.$$

Hence

$$|a'_{j,k+1}, a'_{j+1,k+1}| = |a_{j,k+1}, a_{j+1,k+1}| \geq \bar{d}(a_{j,k+1}, a_{j+1,k+1}) \geq B^{m-k+1}/\lambda.$$

Since γ'_{k+1} is a λ-quasigeodesic with respect to $d_{\mathcal{A}}$ and because of the choice of B, this implies that

$$\bar{d}(a'_{j,k+1}, a'_{j+1,k+1}) \geq B^{m-k+1}/\lambda^2 \geq B^{m-k}$$

Moreover by Lemma 4.8 $a_{j+1,k+1}$ is to the right of $a_{j,k+1}$ on γ_{k+1}. Hence $a'_{j+1,k+1}$ is to the right of $a'_{j,k+1}$ on γ'_{k+1}.

Thus the Claim is proved.

Let $d_{j,k} \in A$ be such that $d_{j,k}$ is equal to the element of A representing the label of any path in D from $a'_{j,k}$ to $a_{j,k+1}$. Thus $|d_{j,k}|_{\mathcal{A}} \leq H$ by the Claim. Then for each $j = 1, \ldots, N+1$ $d_{j,1}, \ldots, d_{j,m-1}$ is a sequence of $m-1$ elements in A of length at most H.

By the choice of N this implies that for some $i < j$ these sequences coincide, that is $d_{j,k} = d_{i,k} = d_k$ for $k = 1, \ldots, m-1$ (see Figure 9).

Let f_k be the element of A represented by the subpath of γ'_k from $a'_{i,k}$ to $a'_{j,k}$. Let b_k be the element of A represented by the subpath of γ'_k from $a'_{j,k}$ to $a'_{i,k}$. Thus $f_k = \phi^{\epsilon_k}(b_k)$ (recall that $t_k = t^{\epsilon_k}$). Let $u_k \in U$ be such that $\alpha[t_k](u_k) = b_k$ and $\alpha[t_k](u_k) = f_k$.

Then $\Sigma = (\underline{p}, \underline{u})$ is an m-annulus, where $\underline{p} = t_1, d_1, t_2, \ldots d_{m-1}, t_m$ and $\underline{u} = u_1, u_2, \ldots, u_m$.

Moreover, Σ is essential. Indeed, suppose $t_{k+1} = t_k^{-1}$ that is $\epsilon_{k+1} = -\epsilon_k$. We may assume that $t_k = 1$ (the case $t_k = -1$ is completely analogous). Thus $f_k, b_{k+1} \in U_1$ and $d_k b_{k+1} d_k^{-1} = f_k$. Recall that $|d_k|_{\mathcal{A}} \leq H$ and $|a'_j, a'_i| \geq |j-i|B \geq B > 3\lambda H$. Hence by Lemma 4.9 $d_k \notin U_1$ and the annulus Σ is essential as required.

Thus we have shown that if $|\gamma_1| \geq 3\lambda B^m(N+1)n$, the conclusion of the Lemma 4.10 holds. Recall that $N = N_0^{m-1}$, $\lambda > 1$, and $B \geq 1$.

Therefore

$$3\lambda B^m(N+1)n \leq (3\lambda B)^m(N_0^{m-1}+1)n \leq (6\lambda B N_0)^m n$$

Thus $E = 6\lambda B N_0$ satisfies the requirement of the Lemma 4.10. □

Lemma 4.11. *Suppose D is a t-minimal diagram as shown in Figure 10, where*

1. *Each path α_k and α'_k is labeled by a $\mathcal{U}_{\pm 1}$-geodesic word.*
2. *The diagrams $T_1, \ldots, T_m$ are t-strips based on $\alpha_1, \ldots, \alpha_m$.*
3. *The only t-relators in D are the t-relators involved in the t-strips $T_1, \ldots, T_m$.*
4. *for each $k < m, k \geq 1$ we have $\bar{d}(u_k, x_{k-1}) \leq C_0 n$, $\bar{d}(v_k, y_{k-1}) \leq C_0 n$.*
5. *we have $|\alpha_1| \geq n(QE)^m$.*

Then there exists an essential m-annulus.

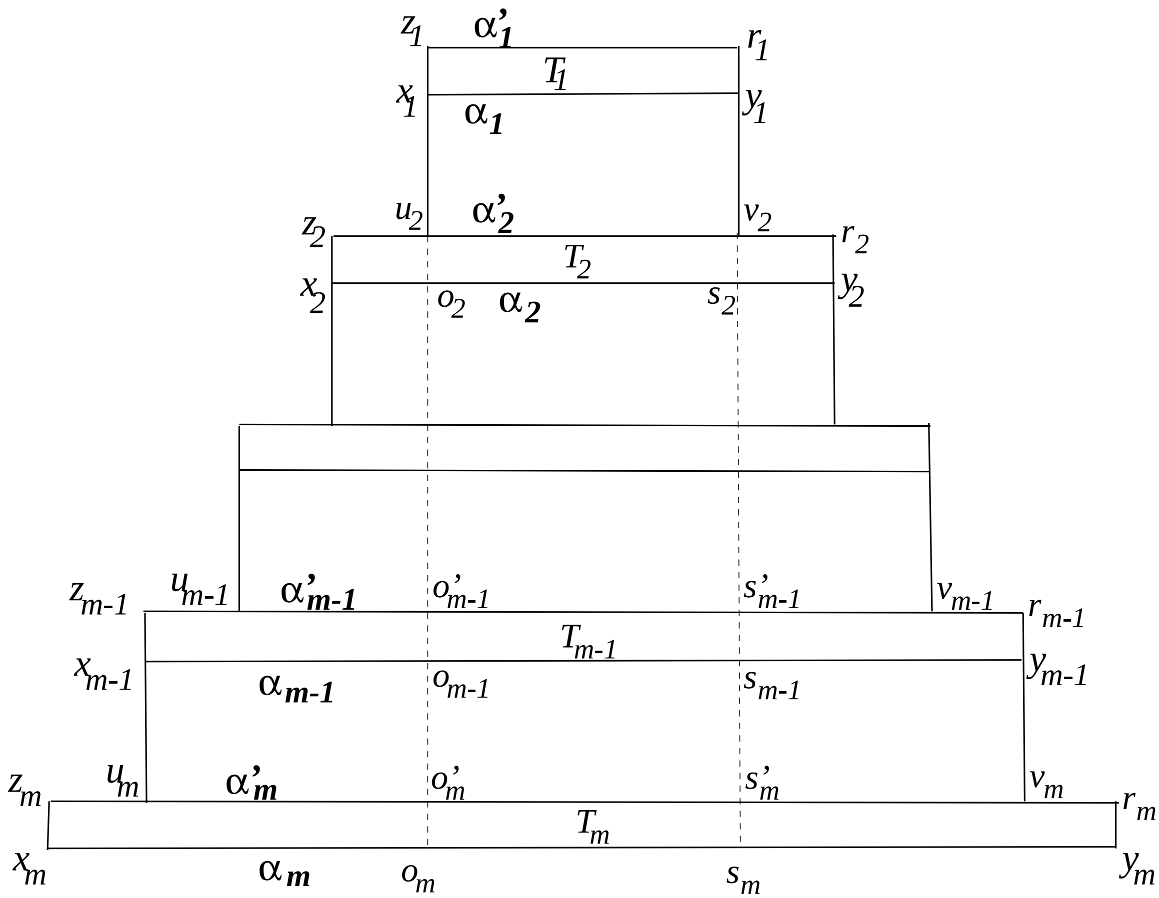

FIGURE 10

Proof. We will show that there exists a t-minimal diagram as in the hypothesis of Lemma 4.10 which by Lemma 4.10 will imply the existence of an essential m-annulus.

Put o_1' and s_1' to be the initial and terminal vertices of α_1' respectively. Put $o_1 = x_1$, $s_1 = y_1$, $o_2' = u_2$ and $s_2' = v_2$.

We define points $o_1', \ldots, o_m'$, $o_1, \ldots, o_m$, $s_1', \ldots, s_m'$ and $s_1, \ldots, s_m$ inductively as follows.

Suppose the vertices o_k', s_k' on α_k' are already defined. Put o_k to be the point on α_k corresponding to o_k' in the k-th t-strip (that is o_k and o_k' are joined by a t-edge). Similarly, put s_k to be the point on α_k corresponding to s_k' in the k-th t-strip. By Lemma 4.7 there exist a vertex o_{k+1}' on α_{k+1}' such that $\bar{d}(o_k, o_{k+1}') \leq C_1 n$. Similarly, by Lemma 4.7 there exist a vertex s_{k+1}' on α_{k+1}' such that $\bar{d}(s_k, s_{k+1}') \leq C_1 n$. This allows us to define all s_i, s_i', o_i, o_i'.

Claim. For each $i = 1, \ldots m$ the vertex s_i' is to the right of o_i' on α_i. Also, for each i we have $\bar{d}(o_i', s_i') \geq nQ^{m-i+1}E^m$.

We will prove this by induction on i.

For $i = 1$ the statement is obvious. Suppose it has been established for $i \geq 1, i < m$. Then $\bar{d}(o_i, s_i) \geq (1\lambda)\bar{d}(o_i', s_i')$. Since $\bar{d}(o_i, o_{i+1}') \leq C_1 n$ and $\bar{d}(s_i, s_{i+1}') \leq C_1 n$, we conclude that $\bar{d}(o_{i+1}', s_{i+1}') \geq (1\lambda)\bar{d}(o_i', s_i') - 2C_1 n \geq (1\lambda)nQ^{m-i+1}E^m - 2C_1 n \geq nQ^{m-i}E^m$.

Also, Lemma 4.7 implies that s_{i+1}' is to the right of o_{i+1}' as required.

Therefore there exists a subdiagram D' of D which satisfies the hypothesis of Lemma 4.10. Namely, D' is obtained from D by removing from D, for $k = 2, \dots, m$ the initial segment of the k-th t-strip T_k ending in (o_k, o'_k) and the terminal segment of the k-th t-strip starting with (s_k, s'_k). By Lemma 4.10 this implies that there exists an essential m-annulus. □

Convention 4.12 (Some terminology and notations)**.** Let α be a nontrivial $\mathcal{A}$-geodesic word and let w be an X-geodesic word such that $\overline{\alpha} = \overline{w}$. Let $n = |w|$.

Note that every $\mathcal{A}$-subword of w is $\mathcal{A}$-geodesic since w is X-geodesic.

Let D be a t-minimal Van-Kampen diagram with the boundary label $w\alpha^{-1}$. Thus D has no t-rings by Lemma 4.2 so that the t-strips in D are situated as shown in Figure 11 (note that there are no t-annuli in D since it is t-minimal).

If we remove the *interiors* of all the t-strips from D, then the remainder $\hat{D}$ of D breaks up into several connected components (which are also simply connected). Denote by D_0 the unique component which contains the path in ∂D labeled by α^{-1}.

We can then form a *dual tree* T_D of D. The vertices of T_D are the connected components of the complement in D of the interiors of t-strips. Two such components are connected by an edge if they border a common t-strips. Note that some of these components may be degenerate (trees or segments).

It is indeed easy to see that T_D is a tree (This can be argued by induction on the number of vertices in T_D.) We say that a component D' has *depth* m if the distance from D_0 to D' in T_D is m. Thus the depth of D_0 is zero.

Every t-strip in D borders exactly two components whose depths differ by one. We call the intersection of this t-strip with the component of higher depth *the upper boundary* of the t-strip. The intersection of the t-strip with the other component is called *the lower boundary* of the t-strip.

Similarly, for every component $D' \neq D_0$ there's a unique component D'' at distance 1 from D' in T_D such that the depth of D'' is smaller than the depth of D'. That is $d(D_0, D') = d(D_0, D'') + 1$ in T_D. Thus D' and D'' border a unique common t-strip. The upper boundary of this t-strip is called *the base* of D. We denote by $\alpha^{D'}$ the word written on the base of D. Thus $\alpha^{D'}$ is a $\mathcal{U}_i$-word.

We also will call the sub-arc in ∂D_0 labeled by α *the base of* D_0.

Since $|w| = n$, the boundary of D_0 can be broken into at most $n+1$ arcs, one of which is labeled α and each of the remaining segments is either labeled by a subword of w or the boundary of one of the t-strips in D adjacent to D_0. We will call this decomposition *the principal decomposition of* ∂D_0.

Similarly, for any component D' of $\hat{D}$ the boundary of D' is broken into at most $n+1$ arcs each of which is either a U-arc or corresponds to a subword of w. We will call this decomposition *the principal decomposition of* $\partial D'$.

For each component D' of D we consider a polygon $P(D')$, based at the identity, with at most $n+1$ sides in the Cayley graph $\Gamma(A, \mathcal{A})$ which are labeled by the same words as the arcs in the principal decomposition of $\partial D'$ (see Figure). Note that for any side of $P(D')$ its label is either a subword of w and thus $\mathcal{A}$-geodesic or a $\mathcal{U}_i$-geodesic word and thus λ-quasigeodesic with respect to $|\cdot|_{\mathcal{A}}$. Thus Lemma 4.5 applies to $P(D')$.

Let β be the $\mathcal{U}_i$-word written on the base of D'. By Lemma the side of $P(D')$ labeled by β can be subdivided into at most n subintervals such that each of them is nC_0-Hausdorff close to a subinterval of one of the remaining sides of $P(D')$. We choose such a subdivision and fix it.

This subdivision induces the corresponding subdivision of the base of D' into at most n subintervals, which we will call *the degeneration subdivision of the base of* D' or sometimes just *the degeneration subdivision for* D'.

Those arcs in the degeneration subdivision of the base of D' which, after passing to $P(D')$, are close to U-arcs in $\partial P(D')$, will be called *U-components* of the base of D'. The remaining arcs of the degeneration subdivision will be called*terminal components* of the base of D'.

Also, let I be a subinterval of an interval I' from the degeneration subdivision of the base of D'. Let I' be C_0n-Hausdorff close to a subinterval J' of one of the remaining sides of $P(D')$. Let J be a subinterval

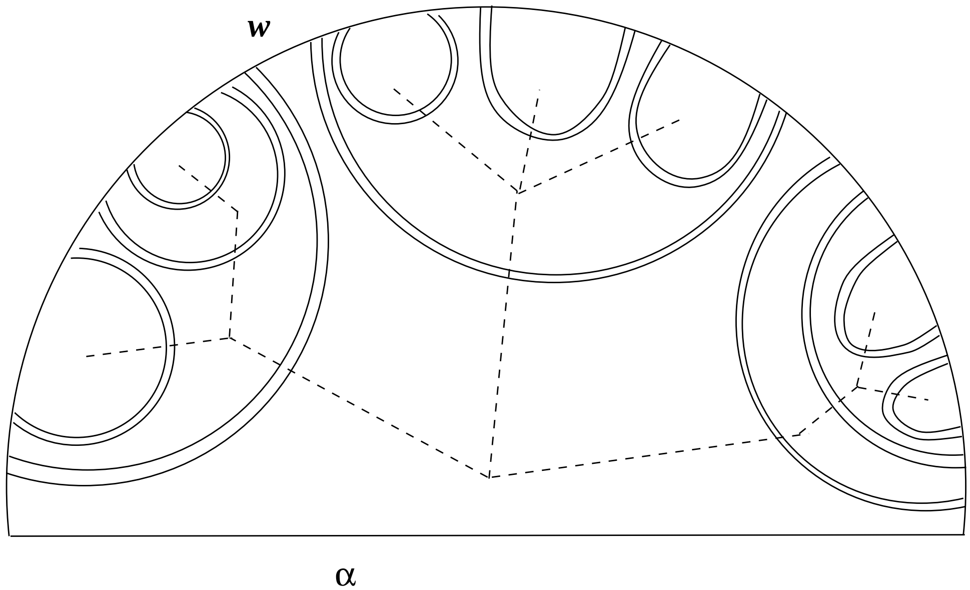

FIGURE 11. The dual tree is drawn in dashed line

of J' such that the distance between the initial points of I and J is at most $C_0 n$ and the distance between the terminal points of I and J is at most $C_0 n$.

Then we say that J is *matched* with I.

Definition 4.13 (Push-out Process)**.** Let $a \in A, a \neq 1$. Let α be an $\mathcal{A}$-geodesic word representing a and let w be an $\mathcal{X}$-geodesic word representing a (recall that $X = A \cup \{t\}$. Put $n = |w| = |a|_X$. Let $L = (nQE)^{M+1}$ and let $F = 4\lambda C_0$.

Consider a t-minimal Van-Kampen diagram D with the boundary label $w\alpha^{-1}$.

Let I be any interval in the degeneration subdivision of α.

We will now describe a process of forming a sequence P of intervals which are members of degeneration subdivisions for some components of $\hat{D}$ (see Figure 12).

Step 0. Put $I_1 = I$ and $P = I_1$.

Step 1.

If I_1 is a terminal component of the degeneration subdivision for D_0, we put $P = I_1$ and terminate the sequence P. Note that in this case there is a subword w_1 of w such that $|I_1| \leq |w_1| + 2C_0$.

Suppose now that I_1 is a U-component of the degeneration subdivision for D_0.

If $|I_1| \leq L$ we put $P = I_1$ and terminate the process.

Assume now that $|I_1| > L$. Then there is a subword J_1 of one of the U-arcs in the boundary of D_0 which is matched with I_1. Thus J_1 is a subword of the lower-boundary label of a certain t-strip in D. Let J_1' be the subword of the upper boundary of this t-strip which corresponds to J_1. Thus J_1' is a subinterval of the base of some component D_1 of $\hat{D}$.

The degeneration subdivision of the base of D_1 has at most n intervals. The degeneration subdivision intervals for D_1 break J_1' into at most n subintervals. Let I_2 be an interval of the biggest length among them. Note that $|J_1| = |J_1'| \leq n|I_2|$ and $|I_1| \leq \lambda(|I_2| + 2C_0) \leq n|I_2|\lambda + 2C\lambda \leq Fn|I_2|$.

If $|I_2| \leq L$ we put $P = I_1, I_2$ and terminate the process. Suppose now that $|I_2| > L$.

Note that I_2 is contained in a unique interval I_2' from the degeneration subdivision of the base of D_1. We now go to Step 2.

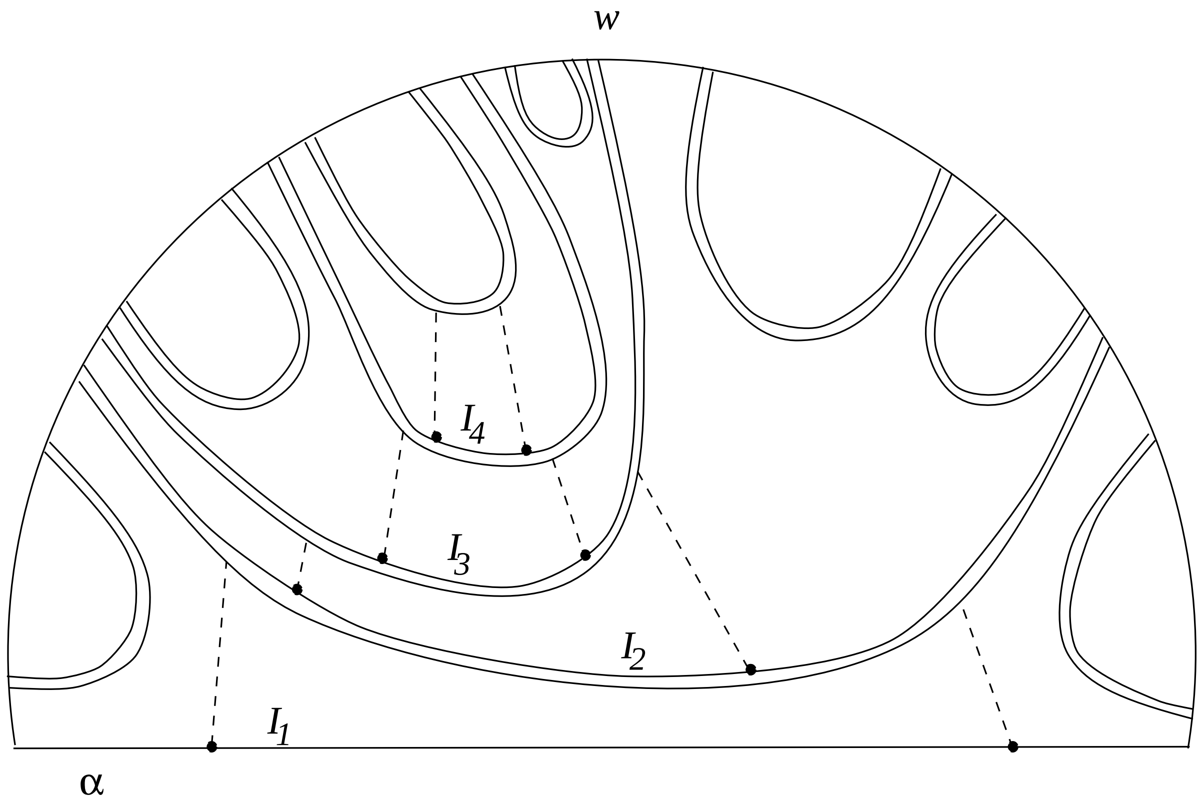

FIGURE 12. The push-out process

Step m**.** Suppose that $m \geq 2$ and we have already constructed the sequence $P = I_1, \ldots, I_m$, where I_m is a subinterval of an interval I'_m in the degeneration subdivision of the base of a component D_m which has depth m and where $|I_m| > L$

Suppose that I'_m is a terminal component of the degeneration subdivision for D_m. Then put $P = I_1, \ldots, I_m$ and terminate the process.

Suppose now that I'_m is a U-component.

Then there is a subword J_m of one of the U-arcs in the boundary of D_m which is matched with I_m. Thus J_m is a subword of the lower-boundary label of a certain t-strip in D. Let J'_m be the subword on the upper boundary of this t-strip which corresponds to J_m. Thus J'_m is a subinterval of the base of some component D_{m+1} of $\hat{D}$.

The degeneration subdivision of the base of D_{m+1} has at most n intervals. The degeneration subdivision intervals for D_{m+1} break J'_m into at most n subintervals. Let I_{m+1} be an interval of the biggest length among them. Note that $|J_m| = |J'_m| \leq n|I_{m+1}|$ and $|I_m| \leq \lambda(n|I_{m+1}| + 2C) \leq Fn|I_{m+1}|$.

If $|I_{m+1}| \leq L$ we put $P = I_1, I_2, \ldots, I_m, I_{m+1}$ and terminate the process.

Suppose now that $|I_{m+1}| > L^{M+1}$. Note that I_{m+1} is contained in a unique interval I'_{m+1} from the degeneration subdivision of the base of D_{m+1}. We now go to Step $m + 1$.

Lemma 4.14.

1. *The push-out process terminates in at most* $M+1$ *steps. That is, if* $P = I_1, \ldots, I_k$ *then* $k \leq M+1$.
2. $|I_1| \leq F^j n^j |I_j|$ *for any* $j > 1$.

3. *Either* $|I_1| \le n^M F^M L$ *or there is a subword* v *of* w *such that* $|I_1| \le n^{M+2} F^{M+2} |v|$.
4. $|I_1| \le 2Ln^{M+3}F^{M+2} = 2LF^{M+2}|w|^{M+3} \le 2(nEQ)^{M+2}F^{M+2}|w|^{M+3} = 2(EQF)^{M+2}|w|^{2M+5}$.

Proof. Part (2) easily follows by induction on j from the description of the push-out process.

Part (1) follows immediately from Lemma 4.11.

To see that part (3) holds, assume that the push-out process terminates after $j \le M+1$ steps with $P = I_1, \dots, I_j$.

There are two possibilities as to why the push-out process stopped.

Case 1. We have $|I_j| \le L$.

Therefore by (2) $|I_1| \le F^j n^j |I_j| \le F^j n^j L \le F^M n^M L$, as required.

Case 2. I_j is contained in I_j' which is a terminal component for the degeneration subdivision of the base of D_j.

In this case there is a subword v of w such that $|I_j| \le \lambda(|v| + 2C_0)$. Therefore

$$|I_1| \le F^j n^j |I_j| \le F^j n^j \lambda(|v| + 2C_0) \le F^j n^j F n |v| \le F^{M+2} n^{M+2} |v|$$

as required.

To see (4), recall that $n = |w|$. Therefore from (3)

$$|I_1| \le n^{M+2}F^{M+2}|w| + n^M F^M L \le Ln^{M+2}F^{M+2}n + Ln^{M+2}F^{M+2} \le 2Ln^{M+3}F^{M+2},$$

since $L \ge 1$. □

Proposition 4.15 (c.f. Theorem 3.8)**.** *Let* $G = \langle A, t | t^{-1}U_{-1}t = U_1 \rangle$ *be an acylindrical HNN-extension, where* A *is word-hyperbolic and* U_{-1}, U_1 *are quasiconvex subgroups of* A.

Then G *is word-hyperbolic and* A *is quasiconvex in* G.

Proof. Let $M \ge 1$ be such that there exist no essential M-annuli for the HNN-presentation of G.

Let $a \in A, a \neq 1$ and let α be an $\mathcal{A}$-geodesic word representing a. Let w be an X-geodesic word representing a and let D be a t-minimal Van-Kampen diagram with the boundary label $w\alpha^{-1}$.

Put $\hat{D}$ to be the complement of the interiors of all the t-strips in D. Let D_0 be the component of $\hat{D}$ containing α. Note that α is the base of D_0.

Denote $n = |w| = |a|_X$. Consider the degeneration subdivision of the base α of D_0. This subdivision contains at most n subintervals. Each of these subintervals has length at most $2(EQF)^{M+2}|w|^{2M+5}$ according to Lemma 4.14. Therefore the length of α is bounded by

$$|a|_A = |\alpha| \le n2(EQF)^{M+2}|w|^{2M+5} = 2(EQF)^{M+2}|w|^{2M+6} = 2(EQF)^{M+2}|a|_X^{2M+6}$$

Thus A has a polynomial distortion function in G and therefore A is quasiconvex in G by Proposition 2.6. □

5. Approximating the word-metric for acylindrical splittings

In this section we show that the word metric on the fundamental group G of a graph of groups can be approximated by normal forms with respect to this graph of groups, provided all the vertex groups are known to be quasiconvex in G. In particular, this applies to the case when the graph of groups is acylindrical.

Let $\mathbb{A}$ be a finite graph of groups where all the vertex groups are word-hyperbolic and all the edge monomorphisms are quasi-isometric embeddings. Let T be a maximal subtree in the underlying graph A of $\mathbb{A}$. For each vertex $v \in VA$ fix a finite generating set X_v of A_v. Put $E = E^+A$ to be the set of positively oriented edges of A.

Then the fundamental group G of the graph of groups $\mathbb{A}$ is generated by the set

$$X = E \cup \bigcup_{v \in VA} X_v.$$

Moreover, G has the following presentation:

$$G = \frac{F(E) * (\underset{v \in VA}{*} A_v)}{e = 1 \text{ for } e \in ET, \alpha[e](u) = e\omega[e](u)e^{-1} \text{ for } e \in E, u \in A_e}$$

Recall that a sequence

$$\sigma = a_0, e_0, a_1, e_1, \ldots, e_{k-1}, a_k$$

is said to be *a path* with respect to $\mathbb{A}$ *from* $\alpha(e_0)$ *to* $\omega(e_{k-1})$ if

(1) each $e_i \in E^{\pm 1}$ and the sequence $e_0, e_1, \ldots, e_{k-1}$ is an edge-path in the underlying graph A and
(2) each $a_i \in A_v$ where v is the initial vertex of e_i for $i \le k-1$ and where v is the terminal vertex of e_{k-1} for $i = k$.

Such σ is called a *reduced path* with respect to $\mathbb{A}$ if, in addition, whenever $e_{i-1} = e_i^{-1}$, we have $a_i \notin \alpha[e_i](A_{e_i})$.

Proposition 5.1. *Suppose that G is word-hyperbolic and all vertex groups A_v are quasiconvex in G. Let p be a vertex of A. Then there exists a constant $K > 0$ with the following property.*

For any $g \in G$ there is a (K, K)-quasigeodesic with respect to d_X word W representing g of the form

$$W = V_1 \ldots V_s$$

where each V_k is either $e^{\pm 1}$ for some $e \in E$ or V_k is a d_{Z_v}-geodesic word for some $v \in VA$ and

$$\overline{V}_1, \ldots, \overline{V}_s$$

is a reduced path from p to p with respect to the graph of groups $\mathbb{A}$.

Proof. Let W be an X-geodesic word representing g. We will transform W to the required form in several steps.

Step 1 Write W as $W = Q_1 \ldots Q_m$ where each Q_i is either an X_v-geodesic word for some $v \in VA$ or $Q_i \in E^{\pm 1}$. Now between every Q_k, Q_{k+1} representing elements of vertex groups v_k and v_{k+1} of A we insert the reduced edge-path r_k in T from v_k to v_{k+1}. Between every Q_k, Q_{k+1} such that $\overline{Q}_k \in A_{v_k}$ and $Q_{k+1} = e \in E^{\pm 1}$ we insert the reduced edge-path r_k in T from v_k to the initial vertex of e. Between every Q_k, Q_{k+1} such that $Q_k = e \in E^{\pm 1}$ and $\overline{Q}_{k+1} \in A_{v_{k+1}}$ we insert the reduced edge-path r_k in T from the terminal vertex of e to v_k. Between every Q_k, Q_{k+1} such that $Q_k \in E^{\pm 1}$ and $\overline{Q}_{k+1} \in E^{\pm 1}$ we insert the reduced edge-path r_k in T from the terminal vertex of Q_k to the initial vertex of Q_{k+1}. We put r_0 to be the reduced edge-path in T from p to the initial vertex of Q_1 when Q_1 is an edge of A and we put r_0 to be the reduced edge-path in T from p to the vertex v_1 when $\overline{Q}_1 \in A_{v_1}$. Analogously, we put r_m to be the reduced edge-path in T from the terminal vertex of Q_m to p when Q_m is an edge of A and we put r_m to be the reduced edge-path in T from v_m to p when $\overline{Q}_m \in A_{v_m}$ Then

$$r_0, \overline{Q}_1, r_1, \ldots r_{m-1}, \overline{Q}_m, r_m$$

is a loop at p in the graph of groups $\mathbb{A}$ which represents g. Observe that each edge-path r_i has length at most N_0 where N_0 is the number of oriented edges of T since T is a tree and r_i has no backtrackings. Moreover, $\overline{r}_i = 1$ in G. Therefore by Lemma 3.4 of [K97] the word

$$W_1 = r_0 Q_1 r_2 \ldots r_{m-1} Q_m r_m$$

is an (L, L)-quasigeodesic with respect to d_X-metric where L is some constant independent of g.

Step 2.

We can write W_1 as $W_1 = U_1 \ldots U_s$ where for $k = 1, \ldots, s$

1. each U_k is either an edge of A or it is a loop in the graph of groups $\mathbb{A}$ representing an element of a vertex group of A and
2. whenever $1 \le i < j \le s$, if the word $U_i \ldots U_j$ (after replacing the subwords in the generating sets of vertex groups by the corresponding elements of the vertex groups) represents a loop in $\mathbb{A}$ at some vertex $v \in VA$, then $U_i \ldots U_j$ does not represent an element of A_v.

Notice that we do not claim that each U_i is a word in generators of some vertex group. Note also that each U_i is (L, L)-quasigeodesic with respect to d_X.

Recall that all vertex groups are quasiconvex in G, so that d_{X_v}-geodesics are d_X-quasigeodesics. For every U_i representing an element of a vertex group A_v put V_i to be an X_v-geodesic word representing the same element as the word U_i. For every U_i which is an edge of A put $V_i = U_i$. Let $W_2 = V_1 \dots V_s$. Thus we have replaced some subwords of an L-quasigeodesic word W_1 by quasigeodesic segments representing the same group elements. By Lemma 3.1 of [K97] the word W_2 is (K, K)-quasigeodesic in the d_X-metric for some constant K independent of g.

Moreover, since each V_i corresponds to a path in $\mathbb{A}$ with the same endpoints as U_i, the word W_2 still corresponds to a path in $\mathbb{A}$ from p to p. The choice of V_i also implies that this path is reduced with respect to $\mathbb{A}$. Thus the word $W_2 = V_1 \dots V_s$ satisfies all the requirements of Proposition 5.1. □

By Theorem 3.10 this immediately implies the following.

Corollary 5.2. *Suppose that $G = \pi_1(\mathbb{A}, T)$ where $\mathbb{A}$ is a finite acylindrical graph of groups and where T is a maximal tree in A. Suppose also that all vertex groups are word-hyperbolic and all edge monomorphisms are quasi-isometric embeddings. Let $X = E^+A \cup \underset{v\in VA_v}{X}$ where X_v is a finite generating set for A_v. Let p be a vertex of A. Then there exists a constant $K > 0$ with the following property.*

For any $g \in G$ there is a (K, K)-quasigeodesic with respect to d_X word W representing g of the form

$$W = V_1 \dots V_s$$

where each V_k is either $e^{\pm 1}$ for some $e \in E$ or V_k is a d_{X_v}-geodesic word for some $v \in VA$ and

$$\overline{V}_1, \dots, \overline{V}_s$$

is a reduced path from p to p with respect to the graph of groups $\mathbb{A}$.

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