Problem 1. Let $X = \{1, 2, 3, 4, 5\}$. For each of the following families of subsets of $X$ decide whether or not this family is a topology on $X$.

1. $T_1 = \{\emptyset, X, \{1, 3\}\}$.
2. $T_2 = \{\emptyset, X, \{1, 3\}, \{2\}\}$.
3. $T_3 = \{\emptyset, X, \{1, 3\}, \{2, 3, 4, 5\}\}$.
4. $T_4 = \{\emptyset, X, \{1, 3\}, \{2, 4, 5\}\}$.

Solution.

1. Yes, $T_1$ is a topology on $X$.
2. No, $T_2$ is not a topology on $X$. Indeed, $\{1, 3\}, \{2\} \in T_2$ but $\{1, 3\} \cup \{2\} = \{1, 2, 3\} \notin T_2$.
3. No, $T_3$ is not a topology on $X$. Indeed, $\{1, 3\}, \{2, 3, 4, 5\} \in T_3$ but $\{1, 3\} \cap \{2, 3, 4, 5\} = \{3\} \notin T_3$.
4. Yes, $T_4$ is a topology on $X$.

Problem 2. Let $X$ be a nonempty set and let $B'$ be a family of subsets of $X$ such that $\cup_{B \in B'} B = X$.

Put $B = \{B_1 \cap \cdots \cap B_n | n \geq 1 \text{ is an integer, and } B_i \in B' \text{ for } i = 1, \ldots, n\}$.

Prove that there exists a topology $T$ on $X$ such that $B$ is a basis for $T$.

Solution. We need to verify that $B$ is a topological basis for $X$. First note that, by using $n = 1$, we see that $B' \subseteq B$, that is for every $B \in B'$ we have $B \in B$.

Since by assumption $\cup_{B \in B'} = X$, it then follows that $\cup_{B \in B} = X$.

Now suppose that $D_1, D_2 \in B$ and $x \in X$ are such that $x \in D_1 \cap D_2$.

By definition of $B$, there exist $m > n \geq 1$ and $B_1, \ldots, B_n, B_{n+1}, \ldots, B_m \in B'$ such that $D_1 = B_1 \cap \cdots \cap B_n$ and $D_2 = B_{n+1} \cap \cdots \cap B_m$.

Put $B = D_1 \cap D_2 = B_1 \cap \cdots \cap B_n \cap B_{n+1} \cap \cdots \cap B_m$.

Then $B \in B$ and $x \in B \subseteq D_1 \cap D_2$.

Therefore $B$ is a topological basis for $X$, as required.

Problem 3.

Let $X = \mathbb{Z}$ and let $T = T_{\text{digital}}$ be the digital topology on $X$.

Put $A = \{n \in \mathbb{Z} | n \text{ is odd}\}$. Compute $\text{Int}(A)$ and $\text{Cl}(A)$ in $(X, T)$.

Solution.

Recall that the digital topology $T$ on $\mathbb{Z}$ has a basis $B = \{A(n) | n \in \mathbb{Z}\}$ where

$$A(n) = \begin{cases} \{n\}, & \text{ if } n \in \mathbb{Z} \text{ is odd} \, , \\ \{n - 1, n, n + 1\}, & \text{ if } n \in \mathbb{Z} \text{ is even} \, . \end{cases}$$

Therefore for every odd integer $n$ the set $\{n\}$ is open in $T$. Hence $A = \cup_{n \text{ odd}} \{n\}$ is also open, as the union of a family of open sets. Thus $\text{int}(A) = A$.

We claim that $\text{Cl}(A) = \mathbb{Z}$. Recall that $\text{Cl}(A) = A \cup A'$ where $A'$ is the set of limit points of $A$. Thus to show that $\text{Cl}(A) = \mathbb{Z}$ it suffices to verify that every even integer is a limit point for $A$.

Suppose $n \in \mathbb{Z}$ is an arbitrary even integer and let $U \in T$ be an open set containing $n$. Since $U$ is a union of sets from the basis $B$, and the only set in $B$
containing $n$ is $A(n)$, it follows that $A(n) = \{n-1, n, n+1\} \subseteq U$. The number $n-1$ is odd, so that and $n-1 \in U \cap (A \setminus \{n\})$. Thus for every open set $U$ containing $n$ we have $U \cap (A \setminus \{n\}) \neq \emptyset$. Therefore, by definition, $n$ is a limit point of $A$. Thus we have established that every even integer is a limit point of $A$ and therefore

$$Cl(A) = A \cup A' = \mathbb{Z}.$$ 

Finally,

$$\partial A = Cl(A) \setminus Int(A) = \mathbb{Z} \setminus A = \{n \in \mathbb{Z}| n \text{ is even}\}.$$ 

**Problem 4.**

Consider $X = \mathbb{R}^2$. Let $\mathcal{T}_v$ be the “vertical interval” topology on $X$ (defined in Exercise 1.19 on p. 37 in the book), and let $\mathcal{T}_e$ be the Euclidean topology on $X$.

Give an example of a point $x \in X$ and a sequence $x_n \in X$ (where $n = 1, 2, 3, \ldots$) such that the sequence $x_n$ converges to $x$ in $(X, \mathcal{T}_e)$ but that $x_n$ does not converge to $x$ in $(X, \mathcal{T}_v)$. Justify that your example has the required properties.

**Solution.**

Let $x = (0, 0)$ and $x_n = (\frac{1}{n}, 0)$ for $n = 1, 2, \ldots$. Then it is obvious that $\lim x_n = x$ in the Euclidean topology $(X, \mathcal{T}_e)$.

We claim that $x_n$ does not converge to $x$ in $(X, \mathcal{T}_v)$. Indeed, take

$$U = \{(0, t): |t| < 1\}.$$ 

Then $U$ is open in $\mathcal{T}_v$, since $U$ is one of the basis elements for $\mathcal{T}_v$. We also have $x = (0, 0) \in U$. However, there does not exist $n \geq 1$ such that $x_n \in U$. Therefore $x_n$ does not converge to $x$ in $(X, \mathcal{T}_v)$.

**Problem 5.**

Consider $X = \mathbb{R}$ and let $\mathcal{T}$ be the upper limit topology on $X$ (defined on p. 31 of the book).

(1) For $A = \{\frac{1}{n}| n \in \mathbb{Z}, n \geq 1\}$ determine whether or not 0 is a limit point of $A$ in $(X, \mathcal{T})$.

(2) Determine whether or not $A$ is a closed set in $(X, \mathcal{T})$.

**Solution.**

(1) Put $U = (-1, 0]$. Then $0 \in U$ and the set $U$ is open in the upper limit topology $\mathcal{T}$, but

$$U \cap (A \setminus \{0\}) = \emptyset.$$ 

Therefore 0 is not a limit point of $A$.

(2) We claim that $A$ is closed in $(X, \mathcal{T})$. To prove this, we will show that the set $V = \mathbb{R} \setminus A$ is open in $(X, \mathcal{T})$.

Let $x \in V$ be arbitrary.

If $x \leq 0$ then choose $y < x \leq 0$ and put $U_x = (y, x]$. Then $x \in U_x \subseteq V$ and $U_x$ is open in $\mathcal{T}$.

If $x \geq 1$ then $x > 1$ (since $x \in V = \mathbb{R} \setminus A$ and $1 \in A$). Choose $y$ so that $1 < y < x$ and put $U_x = (y, x]$. Then again we have $x \in U_x \subseteq V$ and $U_x$ is open in $\mathcal{T}$.

Suppose now that $0 < x < 1$. Since $x \in V = \mathbb{R} \setminus A$, there exists a unique integer $n \geq 1$ such that $\frac{1}{n+1} < x < \frac{1}{n}$. Again choose $y$ so that $\frac{1}{n+1} < y < x$ and put $U_x = (y, x]$. Then again we have $x \in U_x \subseteq V$ and $U_x$ is open in $\mathcal{T}$. 

Thus for every $x \in V$ there exists an open set $U_x$ such that $x \in U_x \subseteq V$. Hence $V = \cup_{x \in V} U_x$ and therefore $V$ is open, as the union of a family of open sets. Since $V = \mathbb{R} \setminus A$, it follows that the set $A$ is closed.

**Problem 6.**
Consider the genotype $w = AAUGCAGGCCACUCU$

(1) Compute the bonding diagram corresponding to $w$.

(2) Let $w' = AAAGCAGGCCACUCU$ (note that $w'$ is obtained from $w$ by changing the third letter of $w$ from $U$ to $A$). Determine whether or not $w'$ and $w$ belong to the same neural network.

**Solution.**

(1) The folded form of $w$ is shown in Figure 1. The corresponding bounding diagram for $w$ is shown in Figure 2.

(2) The folded form of $w'$ is shown in Figure 3.

From here we see that $w$ and $w'$ have different bonding diagrams, and therefore $w$ and $w'$ belong to different neural networks.