Stokes Theorem for manifolds and its classic analogs

1. Stokes Theorem for manifolds.

Definition. A smooth \(n\)-manifold-with-boundary \(M\) is called compact if it can be covered by a finite number of singular \(n\)-cubes, that is, if there exists a finite family \(\gamma_i : [0,1]^n \to M, i = 1, \ldots, k\), of smooth \(n\)-cubes in \(M\) such that

\[
M = \bigcup_{i=1}^{k} \gamma_i([0,1]^n).
\]

Facts.

1. It is known that if \(M \subseteq \mathbb{R}^d\) is closed (that is, if the complement \(\mathbb{R}^d - M\) is an open subset of \(\mathbb{R}^d\)) and bounded (that is \(M\) is contained in some ball in \(\mathbb{R}^d\) of finite radius \(R > 0\) centered around the origin), then \(M\) is compact.

2. If \(M\) is a compact \(n\)-manifold-with-boundary then \(\partial M\) is a compact \((n-1)\)-manifold. Moreover, in this case, for any smooth \(n\)-form \(\alpha\) on \(M\) the integral

\[
\int_M \alpha
\]

exists and is finite.

3. Many compact manifolds-with-boundary arise from the Implicit Function Theorem:
   Let \(F : \mathbb{R}^{n+1} \to \mathbb{R}\) be a smooth function, let \(M = \{x \in \mathbb{R}^{n+1} : F(x) = 0\}\) and supposed that for every \(p \in M\) the derivative \(DF|_p = (\frac{\partial F}{\partial x^1}|_p, \ldots, \frac{\partial F}{\partial x^{n+1}}|_p) \neq (0, \ldots, 0)\). Let \(N = \{x \in \mathbb{R}^{n+1} : F(x) \leq 0\}\). Then \(N\) is an orientable \((n+1)\)-manifold-with-boundary and \(M = \partial N\). If \(N\) is given the standard orientation from \(\mathbb{R}^{n+1}\) and if \(M = \partial N\) is given the induced orientation, then for every \(p \in M\) the outward union normal \(n|_p\) can be computed as:

\[
n|_p = \frac{DF|_p}{\|DF|_p\|} = \frac{1}{\sqrt{\sum_{i=1}^{n} (\frac{\partial F}{\partial x^i}|_p)^2}} (\frac{\partial F}{\partial x^1}|_p, \ldots, \frac{\partial F}{\partial x^{n+1}}|_p)
\]

If, in addition, \(N\) is bounded, then both \(M\) and \(N\) are compact.

For example, the above situation applies to the case of the \(n\)-sphere and \((n+1)\)-ball:

\[
M = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}
\]

and

\[
N = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 \leq 1\},
\]

with \(F(x^1, \ldots, x^{n+1}) = (x^1)^2 + \cdots + (x^{n+1})^2 - 1\).
(4) If $M \subseteq \mathbb{R}^n$ is an $n$-manifold-with-boundary (where $\mathbb{R}^n$ is considered with coordinates $(x^1, \ldots, x^n)$) and if $\omega = f(x) \, dx^1 \wedge \cdots \wedge dx^n$ is an $n$-form on $M$ then
\[
\int_M \omega = \int_M \cdots \int_M f(x) \, dx^1 \cdots dx^n
\]
where the latter integral is the standard integral (in the sense of Calculus III) of a function of several variables.

**Stokes Theorem for manifolds.** Let $M$ be an oriented compact smooth $n$-manifold-with-boundary $M$. Let $\partial M$ be given the induced orientation from $M$. Then for any smooth $(n - 1)$-form $\omega$ on $M$ we have
\[
\int_{\partial M} \omega = \int_M d\omega.
\]

2. Classic analogs of Stokes’ Theorem

There are three classic analogs of Stokes’ Theorem for manifolds that can all be derived from Stokes’ Theorem for manifolds (even though historically they were proved first).

**A. Green’s Theorem.** Let $M \subseteq \mathbb{R}^2$ be a compact smooth 2-manifold-with-boundary. The manifold $M$ is given the standard orientation from $\mathbb{R}^2$. Then $\partial M$ is a smooth compact 1-manifold with orientation induced from $M$ (note that we allow for the possibility that $\partial M$ consists of several components).

Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions. Then
\[
\int_{\partial M} \alpha \, dx + \beta \, dy = \int_M \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \, dx \wedge dy = \int \int \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \, dx \, dy.
\]

**Proof.**

Put $\omega = \alpha \, dx + \beta \, dy$. Thus $\omega$ is a smooth 1-form on $M$ and
\[
d\omega = \left( \frac{\partial \alpha}{\partial x} \, dx + \frac{\partial \alpha}{\partial y} \, dy \right) \wedge dx + \left( \frac{\partial \beta}{\partial x} \, dx + \frac{\partial \beta}{\partial y} \, dy \right) \wedge dy = \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \, dx \wedge dy.
\]

By Stokes’ Theorem
\[
\int_{\partial M} \alpha \, dx + \beta \, dy = \int_{\partial M} \omega = \int_M d\omega = \int_M \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \, dx \wedge dy.
\]

**B. The Divergence Theorem.**

Let $F = (\alpha, \beta, \gamma) : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function. (In the classic terminology such a function $F$ is often referred to as a vector field on $\mathbb{R}^3$. We will avoid this terminology since for us the term “vector field” has specific
technical meaning in the context of smooth manifolds). The divergence $\text{div} F$ of $F$ is defined as

$$\text{div}(F) = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}.$$ 

The Divergence Theorem states the following:

Let $M \subseteq \mathbb{R}^3$ be a smooth 3-manifold with boundary given the standard orientation from $\mathbb{R}^3$. Let $\partial M$ be given the induced orientation from $M$. We equip both $M$ and $\partial M$ with the Riemannian metrics obtained by restricting the standard inner product in $\mathbb{R}^3$ to their tangent spaces. Let $F = (\alpha, \beta, \gamma) : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function. Let $dV$ be the volume form on $M$ (which is just the restriction of the standard volume form $dx \wedge dy \wedge dz$ from $\mathbb{R}^3$) and let $dA$ be the area form for $\partial M$. Let $n$ be the outward unit normal for $\partial M$.

Then

$$\int_M \text{div} F \, dV = \int_{\partial M} \langle F, n \rangle \, dA$$

and this equality can be re-written as

$$\int_M \left( \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) \, dx \wedge dy \wedge dz = \int_{\partial M} (n^1 \alpha + n^2 \beta + n^3 \gamma) \, dA.$$

Proof.

Define $\omega$ on $M$ as $\omega = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$. A direct computation then shows that $d\omega = \text{div}(F) \, dx \wedge dy \wedge dz = \text{div}(F) \, dV$. Moreover, on $\partial M$ we have $n^1 dA = dy \wedge dz$, $n^2 dA = dz \wedge dx$, and $n^3 dA = dx \wedge dy$. Hence on $\partial M$ we have

$$\langle F, n \rangle \, dA = (n^1 \alpha dA + n^2 \beta dA + n^3 \gamma dA) = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy.$$

Hence by Stokes Theorem for manifolds:

$$\int_M \text{div} F \, dV = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \langle F, n \rangle \, dA.$$

C. The classic Stokes’ Theorem.

For $\mathbb{R}^3$ with coordinates $(x, y, z)$ we will use the symbolic notation $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. For a function $F = (\alpha, \beta, \gamma) : \mathbb{R}^3 \to \mathbb{R}^3$, denote

$$\text{curl} F := \nabla \times F = \left( \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right).$$

The classic Stokes’ Theorem state the following:

Let $M \subseteq \mathbb{R}^3$ be an oriented 2-manifold-with-boundary with outward normal $n$. Let $\partial M$ be given the induced orientation from $M$ (thus $\partial M$ is an oriented 1-manifold, possibly consisting of several components). Let let $T = (T^1, T^2, T^3)$ be the positively oriented unit tangent to $M$ (so that if $ds$ is the volume form for $\partial M$ then $ds(T) = 1$). Let $dA$ be the area form for $M$. Let $F = (\alpha, \beta, \gamma) : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function.
Then:

$$\int_M \langle \text{curl } F, \mathbf{n} \rangle \, dA = \int_{\partial M} \langle F, \mathbf{T} \rangle \, ds$$

and this equality can be rewritten as

$$\int_{\partial M} \alpha \, dx + \beta \, dy + \gamma \, dz = \int_M \left( n^1 \left( \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) + n^2 \left( \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) + n^3 \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \right) \, dA.$$

**Proof.**

Consider the 1-form $\omega$ on $M$ defined as $\omega = \alpha \, dx + \beta \, dy + \gamma \, dz$. A direct computation shows that

$$d\omega = \left( \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) dy \wedge dz + \left( \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) dz \wedge dx + \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx \wedge dy.$$

We have

$$\langle \text{curl } F, \mathbf{n} \rangle = n^1 \left( \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) + n^2 \left( \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) + n^3 \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right).$$

Since on $M$ we have $n^1 \, dA = dy \wedge dz$, $n^2 \, dA = dz \wedge dx$ and $n^3 \, dA = dx \wedge dy$,

it follows that on $M$

$$\langle \text{curl } F, \mathbf{n} \rangle \, dA = \left( \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) dy \wedge dz + \left( \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) dz \wedge dx + \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx \wedge dy = d\omega.$$

Also, since $ds(\mathbf{T}) = 1$, on $\partial M$ we have $T^1 \, ds = dx$, $T^2 \, ds = dy$ and $T^3 \, ds = dz$. This implies that on $\partial M$

$$\langle F, \mathbf{T} \rangle \, ds = \alpha T^1 \, ds + \beta T^2 \, ds + \gamma T^3 \, ds = \alpha dx + \beta dy + \gamma dz = \omega.$$

Therefore, by the Stokes’ Theorem for manifolds,

$$\int_M \langle \text{curl } F, \mathbf{n} \rangle \, dA = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \langle F, \mathbf{T} \rangle \, ds.$$