H/work 10 (Solutions).

1. Consider the following vector fields on \(\mathbb{R}^3\) (taken with coordinates \((x, y, z)\):

\[
X = 2xy \frac{\partial}{\partial x} + (z^2 + x^2) \frac{\partial}{\partial z}
\]

and

\[
Y = e^{xy} \frac{\partial}{\partial x} + \cos(2xz) \frac{\partial}{\partial y}
\]

Compute the vector field \([X, Y]\) on \(\mathbb{R}^3\).

Solution.
We have:

\[
[2xy \frac{\partial}{\partial x}, e^{xy} \frac{\partial}{\partial x}] = 2xy \frac{\partial e^{xy}}{\partial x} \frac{\partial}{\partial x} - e^{xy} \frac{\partial 2xy}{\partial x} \frac{\partial}{\partial x} = (2xy^2 - 2y)e^{xy} \frac{\partial}{\partial x},
\]

and

\[
[2xy \frac{\partial}{\partial x}, \cos(2xz) \frac{\partial}{\partial y}] = 2xy \frac{\partial \cos(2xz)}{\partial x} \frac{\partial}{\partial x} - \cos(2xz) \frac{\partial 2xy}{\partial y} \frac{\partial}{\partial x} = -4xyz \sin(2xz) \frac{\partial}{\partial x} - 2x \cos(2xz) \frac{\partial}{\partial x}
\]

and

\[
[(z^2 + x^2) \frac{\partial}{\partial z}, e^{xy} \frac{\partial}{\partial x}] = (z^2 + x^2) \frac{\partial e^{xy}}{\partial z} \frac{\partial}{\partial x} - e^{xy} \frac{\partial (z^2 + x^2)}{\partial z} \frac{\partial}{\partial z} = -2xe^{xy} \frac{\partial}{\partial z},
\]

and

\[
[(z^2 + x^2) \frac{\partial}{\partial z}, \cos(2xz) \frac{\partial}{\partial y}] = (z^2 + x^2) \frac{\partial \cos(2xz)}{\partial z} \frac{\partial}{\partial y} - \cos(2xz) \frac{\partial (z^2 + x^2)}{\partial y} \frac{\partial}{\partial z} = -2x(z^2 + x^2) \sin(2xz) \frac{\partial}{\partial y}
\]

Therefore

\[
[X, Y] = [2xy \frac{\partial}{\partial x}, e^{xy} \frac{\partial}{\partial x}] + [2xy \frac{\partial}{\partial x}, \cos(2xz) \frac{\partial}{\partial x}] + [(z^2 + x^2) \frac{\partial}{\partial z}, e^{xy} \frac{\partial}{\partial x}] + [(z^2 + x^2) \frac{\partial}{\partial z}, \cos(2xz) \frac{\partial}{\partial y}] = ((2xy^2 - 2y)e^{xy} - 2x \cos(2xz)) \frac{\partial}{\partial x} - (4xyz \sin(2xz) + 2x(z^2 + x^2)) \frac{\partial}{\partial y} - 2xe^{xy} \frac{\partial}{\partial z}.
\]

2. Let \(X, Y, Z\) be smooth vector fields on an \(n\)-manifold \(M\). Prove the Jacobi identity:

\[
[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.
\]

Solution
We have

\[
\begin{align*}
[[X,Y],[Z]] + [[Y,Z],X] + [[Z,X],Y] = \\
(XY - YX)Z - Z(XY - YX) + (YZ - ZY)X - X(YZ - ZY) + \\
(XY - YX)Z - Y(ZX - XZ) = \\
XYZ - YZX - ZYX + ZYX - XZY + XZY + \\
+ZXY - XZY - YZX + YXZ = 0.
\end{align*}
\]

3. Recall that a form \( \omega \) is called \textit{closed} if \( d\omega = 0 \), and that \( \omega \) is called \textit{exact} if \( \omega = d\tau \) for some form \( \tau \).

Let \( \omega = \alpha(x,y,z) \, dx \wedge dy + \beta(x,y,z) \, dx \wedge dz + \gamma(x,y,z) \, dy \wedge dz \) be a 2-form on \( \mathbb{R}^3 \).

(a) Write down explicit conditions for the functions \( \alpha, \beta, \gamma \) equivalent to the condition \( d\omega = 0 \).

(b) Write an explicit condition for \( \alpha, \beta, \gamma \) equivalent to saying that \( \omega \) is exact.

(c) Show that the form \( \eta = z^2 \, dx \wedge dy + 2zy \, dx \wedge dz \) is an exact 2-form on \( \mathbb{R}^3 \).

\textbf{Solution.}

(a) We have

\[
\begin{align*}
d\omega &= \left( \frac{\partial \alpha}{\partial x} \, dx + \frac{\partial \alpha}{\partial y} \, dy + \frac{\partial \alpha}{\partial z} \, dz \right) dx \wedge dy + \\
&\quad \left( \frac{\partial \beta}{\partial x} \, dx + \frac{\partial \beta}{\partial y} \, dy + \frac{\partial \beta}{\partial z} \, dz \right) dx \wedge dz + \\
&\quad \left( \frac{\partial \gamma}{\partial x} \, dx + \frac{\partial \gamma}{\partial y} \, dy + \frac{\partial \gamma}{\partial z} \, dz \right) dy \wedge dz = \\
&= \left( \frac{\partial \alpha}{\partial z} \, dx \wedge dy + \frac{\partial \beta}{\partial y} \, dx \wedge dz + \frac{\partial \gamma}{\partial x} \, dy \wedge dz \right) dx \wedge dy \wedge dz.
\end{align*}
\]

Thus \( d\omega = 0 \) if and only if \( \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial x} \equiv 0 \) on \( \mathbb{R}^3 \).

(b) The form \( \omega \) is exact if there exists a 1-form \( \mu = f \, dx + g \, dy + h \, dz \) such that \( d\mu = \omega \). We have

\[
\begin{align*}
d\mu &= \left( \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz \right) \wedge dx + \\
&\quad \left( \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy + \frac{\partial g}{\partial z} \, dz \right) \wedge dy + \\
&\quad \left( \frac{\partial h}{\partial x} \, dx + \frac{\partial h}{\partial y} \, dy + \frac{\partial h}{\partial z} \, dz \right) \wedge dz = \\
&= \left( -\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \right) dx \wedge dy + \\
&\quad \left( -\frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} \right) dx \wedge dz + \\
&\quad \left( -\frac{\partial g}{\partial z} + \frac{\partial h}{\partial y} \right) dy \wedge dz.
\end{align*}
\]
Thus $\omega$ is exact on $\mathbb{R}^3$ if and only if there exist smooth function $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\alpha = -\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}, \quad \beta = -\frac{\partial f}{\partial z} + \frac{\partial h}{\partial x}, \quad \gamma = -\frac{\partial g}{\partial z} + \frac{\partial h}{\partial y}$$
on $\mathbb{R}^3$.

(c) Let $\mu = -yz^2 \, dx$. then

$$d\mu = -(z^2 \, dy + 2yz \, dz) \wedge dx = z^2 \, dx \wedge dz + 2yz \, dy \wedge dz = \eta.$$ Thus $\eta$ is exact, as required.

4.

Let $\gamma : [0, 1]^r \to M$ be an $r$-cube where $r \geq 2$.

(1) For $1 \leq i \leq j \leq r - 1$ compute the face-maps $(\gamma_{(i, \alpha)})(j, \beta)$ and $(\gamma_{(j+1, \beta)})(i, \alpha)$ for $x \in [0, 1]^{r-2}$. Conclude that $(\gamma_{(i, \alpha)})(j, \beta) = (\gamma_{(j+1, \beta)})(i, \alpha)$ for $1 \leq i \leq j \leq r - 1$.

(2) Prove that $\partial \partial \gamma = 0$.

Solution.

(1) Let $1 \leq i \leq j \leq r - 1$.

For $x = (x^1, \ldots, x^{r-2}) \in [0, 1]^{r-2}$ we have:

$$(\gamma_{(i, \alpha)})(j, \beta)(x) = \gamma_{(i, \alpha)}(x^1, \ldots, x^{i-1}, \beta, x^j, \ldots, x^{r-2}) = \gamma(x^1, \ldots, x^{i-1}, \alpha, x^j, \ldots, x^{r-2}),$$and

$$(\gamma_{(j+1, \beta)})(i, \alpha)(x) = \gamma_{(j+1, \beta)}(x^1, \ldots, x^{i-1}, \alpha, x^j, \ldots, x^{r-2}) = \gamma(x^1, \ldots, x^{i-1}, \alpha, x^j, \ldots, x^{r-2}),$$

Thus $(\gamma_{(i, \alpha)})(j, \beta) = (\gamma_{(j+1, \beta)})(i, \alpha)$, as required.

(2) We have

$$\partial \gamma = \sum_{k=1}^{r} \sum_{\alpha=0,1} (-1)^{k+\alpha} \gamma_{(k, \alpha)}$$

and

$$\partial \partial \gamma = \sum_{k=1}^{r} \sum_{\alpha=0,1} \sum_{\ell=1}^{r-1} \sum_{\beta=0,1} (-1)^{k+\ell+\alpha+\beta} \gamma_{(k, \alpha)}(\ell, \beta).$$

Note that for every $(\gamma_{(k, \alpha')})(\ell, \beta')$, where $1 \leq k \leq r$, $1 \leq \ell \leq r - 1$, either $k \leq \ell \leq r - 1$ or $\ell \leq k - 1 \leq r - 1$. Thus either $(\gamma_{(k, \alpha')})(\ell, \beta') = (\gamma_{(i, \alpha)})(j, \beta)$, where $i \leq j \leq r - 1$ and $\alpha' = \alpha$, $\beta' = \beta$, or $(\gamma_{(k, \alpha')})(\ell, \beta') = (\gamma_{(j+1, \beta)})(i, \alpha)$ where $i \leq j \leq r - 1$ and $\alpha' = \beta$, $\beta' = \alpha$.

Hence in the formula for $\partial \partial \gamma$ the terms $(\gamma_{(i, \alpha)})(j, \beta)$, $(\gamma_{(j+1, \beta)})(i, \alpha)$, where $1 \leq i \leq j \leq r - 1$ and $\alpha, \beta \in \{0, 1\}$, are paired-off and they occur with opposite signs. Therefore, by part (1), $\partial \partial \gamma = 0$. 