1. Consider the following vector fields on $\mathbb{R}^3$ (taken with coordinates $(x, y, z)$):

\[ X = 2xy \frac{\partial}{\partial x} + (z^2 + x^2) \frac{\partial}{\partial z} \]

and

\[ Y = e^{xy} \frac{\partial}{\partial x} + \cos(2xz) \frac{\partial}{\partial y}. \]

Compute the vector field $[X, Y]$ on $\mathbb{R}^3$.

2. Let $X, Y, Z$ be smooth vector fields on an $n$-manifold $M$. Prove the Jacobi identity:

\[ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \]

3. Recall that a form $\omega$ is called **closed** if $d\omega = 0$, and that $\omega$ is called **exact** if $\omega = d\tau$ for some form $\tau$.

Let

\[ \omega = \alpha(x, y, z) \, dx \wedge dy + \beta(x, y, z) \, dx \wedge dz + \gamma(x, y, z) \, dy \wedge dz \]

be a 2-form on $\mathbb{R}^3$.

(a) Write down explicit conditions for the functions $\alpha, \beta, \gamma$ equivalent to the condition $d\omega = 0$.

(b) Write an explicit condition for $\alpha, \beta, \gamma$ equivalent to saying that $\omega$ is exact.

(c) Show that the form

\[ \eta = z^2 \, dx \wedge dy + 2zy \, dx \wedge dz \]

is an exact 2-form on $\mathbb{R}^3$.

4. Let $\gamma : [0, 1]^r \to M$ be an $r$-cube where $r \geq 2$.

(1) For $1 \leq i \leq j \leq r - 1$ compute the face-maps $(\gamma(i,\alpha))(j,\beta)(x)$ and $(\gamma(j+1,\beta))(i,\alpha)(x)$ for $x \in [0, 1]^{r-2}$. Conclude that $(\gamma(i,\alpha))(j,\beta) = (\gamma(j+1,\beta))(i,\alpha)$ for $1 \leq i \leq j \leq r - 1$.

(2) Prove that $\partial \partial \gamma = 0$. 

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