We want to define $\int_\gamma \alpha$, for any $r$-form $\alpha$ and singular $r$-cube $\gamma$ in a manifold $M$:

a) **Definition:** In coordinates $(U, (x^1, \ldots, x^n))$ on $M$, we have $\gamma : [0,1]^r \to M, \gamma(u^1, \ldots, u^r) = (x^1(u^1, \ldots, u^r), \ldots, x^n(u^1, \ldots, u^r))$, and $\alpha = \sum_{I=(i_1, \ldots, i_r)} a_I dx^{i_1} \wedge \cdots \wedge dx^{i_r}$. So

$$\gamma^* \alpha = \sum_{I} a_I \left( \sum_{j=1}^{r} \frac{\partial x^{i_j}}{\partial u^n} du^n \right) \wedge \cdots \wedge \left( \sum_{j=1}^{r} \frac{\partial x^{i_r}}{\partial u^n} du^n \right)$$

$$= \sum_{I} a_I \det \frac{\partial(x^{i_1}, \ldots, x^{i_r})}{\partial(u^1, \ldots, u^r)} du^1 \wedge \cdots \wedge du^r$$

So, now that we’ve written $\gamma^* \alpha = f(u^1, \ldots, u^r) du^1 \wedge \cdots \wedge du^r$, just define

$$\int_\gamma \alpha = \int_0^1 \cdots \int_0^1 f(u^1, \ldots, u^r) \, du^1 \cdots du^r.$$

b) **Example:** Line integrals, see Handout #15 d).

c) **Example:** $M = \mathbb{R}^2$. Take the 2-form $\alpha = dx^1 \wedge dx^2$ (the Euclidean “area form”) and the singular 2-cube $\gamma(u^1, u^2) = (9(u^1 - \frac{1}{2})^2, u^2), 0 \leq u^1 \leq 1$. Then

$$\int_\gamma \alpha = 3.$$

d) **Example:** $M = \mathbb{R}^2$, $\alpha = dx^1 \wedge dx^2$, $\gamma(u^1, u^2) = ((u^1 + 1) \cos(2\pi u^2), (u^1 + 1) \sin(2\pi u^2))$.

$\int_\gamma \alpha$ should be $\pi(2)^2 - \pi(1)^2 = 3\pi$. Check this!

e) In d), $\alpha = d(\frac{1}{2}(x^1 dx^2 - x^2 dx^1)) = d\beta$, where $\beta = \frac{1}{2}(x^1 dx^2 - x^2 dx^1)$. So,

$$\int_\gamma \alpha = \int_\gamma d\beta = 3\pi.$$ 

Now, calculate $\int_{\partial \gamma} \beta$. We have

$$\partial \gamma = \gamma_{2,0} + \gamma_{1,1} - \gamma_{2,1} - \gamma_{1,0} = \gamma_{1,1} - \gamma_{1,0},$$

where

$$\gamma_{2,0}(u^1) = (u^1 + 1, 0), \quad \gamma_{1,1}(u^2) = (2 \cos(2\pi u^2), 2 \sin(2\pi u^2)), \quad \gamma_{2,1}(u^1) = (u^1 + 1, 0), \quad \gamma_{1,0}(u^2) = (\cos(2\pi u^2), \sin(2\pi u^2)).$$

Check that the answer is $3\pi$!
Math 481
18. Stokes' Theorem

1. **Stokes' Theorem**: If \( \alpha \) is an \((r-1)\) form on \( M \) and \( \gamma \) is a singular \( r \)-chain, then

\[
\int_{\gamma} d\alpha = \int_{\partial\gamma} \alpha
\]

2. a) An \( r \)-form \( \alpha \) is **closed** if \( d\alpha = 0 \). It is **exact** if \( \alpha = d\beta \) for some \((r-1)\)-form \( \beta \).

b) Notice that if \( \alpha \) is exact, it is closed since \( d^2 = 0 \).

c) A closed form may **not** be exact!

d) Example: \( M = \mathbb{R}^2 - (0,0) \), \( \alpha = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \).

Check that \( d\alpha = 0 \). So \( \alpha \) is a closed 1-form. Is \( \alpha = df \) for some function (i.e. 0-form) \( f \) on \( M - (0,0) \)? No! On any region that is smoothly contractible to a point, \( \alpha = d\theta \) where \( \theta \) is the angle function of polar coordinates. But, \( \theta \) cannot be defined on all of \( M \) (why?)

3. a) **Poincaré Lemma**: If \( M \) is smoothly contractible to a point ("SCP"), then every closed \( r \)-form \( \alpha \) on \( M \) is exact (i.e. \( \alpha = d\beta \)).

b) So, if \( M \) is SCP and \( \alpha \) is a closed \( r \)-form on \( M \), then \( \int_{\gamma} \alpha \) depends only on \( \partial\gamma \! \).

**Reason**: If \( \gamma_1, \gamma_2 \) are \( r \)-chains with \( \partial\gamma_1 = \partial\gamma_2 \),

\[
\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \int_{\gamma_1 - \gamma_2} \alpha
\]

(Poincaré) \( = \int_{\gamma_1 - \gamma_2} d\beta \)

(Stokes) \( = \int_{\partial(\gamma_1 - \gamma_2)} \beta \)

\( = \int_{\partial\gamma_1 - \partial\gamma_2} \beta = 0 \)