16. Singular r-chains

1. a) A "singular r-cube" \( \gamma \) in \( M \) is a smooth map \( \gamma : [0, 1]^r \to M \).
   b) A "singular r-chain" is a formal linear combination of singular r-cubes: \( \sum_{i=1}^{d} a_i \gamma_i \)
   where \( a_i \in \mathbb{R} \).

2. Given a singular 1-cube \( \gamma \) in \( \mathbb{R}^3 \), we want to define the boundary of \( \gamma \) (denoted \( \partial \gamma \)) to be \( \gamma(1) - \gamma(0) \) (a singular 0-chain).

3. Given a singular 2-cube \( \gamma \) in \( \mathbb{R}^3 \), by looking at what \( \gamma \) does to the edges of the square, we get four 1-cubes, each oriented by increasing \( u^i \). We want \( \partial \gamma \) to be
   \[ \gamma_{2,0} + \gamma_{1,1} - \gamma_{2,1} - \gamma_{1,0}. \]

4. For any r-cube \( \gamma : [0, 1]^r \to M \), define:
   a) \( (r-1) \)-cubes \( \gamma_{i,\alpha} : [0, 1]^{r-1} \to M \), \( 1 \leq i \leq r \), \( \alpha = 0 \) or \( 1 \), by
   \[ \gamma_{i,\alpha}(u^1, \ldots, u^{i-1}, u^{i+\alpha-1}) = \gamma(u^1, \ldots, u^{i-1}, \alpha, u^i, \ldots, u^{r-1}), \]
   b) the boundary of \( \gamma \) by
   \[ \partial \gamma = \sum_{i,\alpha} (-1)^{i+\alpha} \gamma_{i,\alpha}. \]
   (If \( r = 0 \), set \( \partial \gamma = 1 \in \mathbb{R} \).)

5. Theorem: For any r-cube \( \gamma \), \( \partial^2 \gamma = 0 \).