1. a) A differential \( r \)-form \( \alpha \) on \( M \) is an antisymmetric \( r \)-covariant tensor field. At each \( p \in M \):
\[
\alpha(\ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots) = -\alpha(\ldots, \vec{v}_j, \ldots, \vec{v}_i, \ldots)
\]
b) Examples
(i) If \((U, (x^1, \ldots, x^n))\) is a patch, \(dx^1, \ldots, dx^n\) are a basis for the 1-forms at any point in \( U \).
(ii) If \( M = \mathbb{R}^3 \), the scalar triple product \( \alpha(\vec{v}, \vec{w}, \vec{u}) = \vec{v} \times \vec{w} \cdot \vec{u} \) is a 3-form on \( M \).

c) An \( r \)-form is determined at any point by its \( \left( \begin{array}{c} n \\ r \end{array} \right) \) components
\[
a_{i_1 < \ldots < i_r} = \alpha \left( \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}} \right).
\]
d) We write
\[
I = (i_1, \ldots, i_r), \quad |I| = r, \quad \frac{\partial}{\partial x^I} = \left( \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}} \right), \quad \rightarrow = (i_1 < \ldots < i_r).
\]
For \( J = (j_1, \ldots, j_r) \), set
\[
\delta_J^I = \begin{cases} 
1, & \text{if } J \text{ is an even permutation of } I \\
-1, & \text{if } J \text{ is an odd permutation of } I \\
0, & \text{otherwise.}
\end{cases}
\]

2. a) The wedge product \( \alpha \wedge \beta \) of an \( r \)-form \( \alpha \) and an \( s \)-form \( \beta \) is defined by
\[
(\alpha \wedge \beta)(\vec{v}_1, \ldots, \vec{v}_{r+s}) = \sum \delta_{(i_1,\ldots,i_r)}^{(K,L)} \alpha(\vec{v}_K) \beta(\vec{v}_L),
\]
where \( \Sigma \) is over all permutations \((K,L)\) of \((1, \ldots, r+s)\).

b) Example:
\[
(dx^1 \wedge dx^2) \left( \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) = \begin{cases} 
1 & \text{if } k = 1, l = 2 \\
-1 & \text{if } k = 2, l = 1 \\
0 & \text{otherwise}
\end{cases}
\]
c) Similarly, \( dx^I \left( \frac{\partial}{\partial x^j} \right) = \delta_I^J \).
d) In particular, \( dx^1 \wedge dx^2 = -dx^2 \wedge dx^1 \) and \( dx^i \wedge dx^j = -dx^j \wedge dx^i \).
3. **Key fact:**

Let \( \alpha \) be an \( r \)-form on \( M \). In a coordinate patch, write \( \alpha(\partial/\partial x^I) = a_I \) for all \( I = i_1 < \ldots < i_r \). Then

\[
\alpha = a_{i_1 < \ldots < i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r} := a_I dx^I.
\]

To prove this, note that the left hand side and the right hand side of the above equation do the same thing to any \( \frac{\partial}{\partial x^J} \) since \( dx^J \left( \frac{\partial}{\partial x^J} \right) = \delta^J_I \). By linearity, then, both sides are the same tensor.

b) From a) we conclude that the \( dx^J = dx^{i_1} \wedge \ldots \wedge dx^{i_r} \), \( i_1 < \ldots < i_r \), form a basis for the \( r \)-forms at a point.

c) Therefore, the \( r \)-forms at a point form a vector space of dimension \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

4. **Properties of \( \wedge \):**

a) **Bilinear:** \((a_1 \alpha_1 + a_2 \alpha_2) \wedge \beta = a_1(\alpha_1 \wedge \beta) + a_2(\alpha_2 \wedge \beta) \) etc.

b) if \( \alpha \) is an \( r \)-form and \( \beta \) is an \( s \)-form, then \( \beta \wedge \alpha = (-1)^r \alpha \wedge \beta \).

c) **Associative:** \((\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = \alpha \wedge \beta \wedge \gamma \).
15. Pullback of Differential Forms

Pullback of an r-form (in fact, of any r-covariant tensor)
Reference: Frankel, pp. 77-82
For a map \( F : M^m \to N^n \), an r-covariant tensor \( \alpha \) on \( N \) pulls back to an r-covariant tensor \( F^* \alpha \) on \( M \):

a) Invariant definition of \( F^* \alpha \) at any \( p \in M \):

\[
(F^* \alpha)(\tilde{\nu}_1, \ldots, \tilde{\nu}_r) = \alpha(F_* \tilde{\nu}_1, \ldots, F_* \tilde{\nu}_r)
\]

i.e., \( \alpha \) pulls back because tangent vectors push forward!

b) But, vector fields on \( M \) do not push forward to vector fields on \( N \). For example, take \( M = \mathbb{R}, N = \mathbb{R}^2 \) and \( \dot{V} = \frac{\partial}{\partial t} \) a vector field on \( M \). Let \( F \) be the map pictured below:

Then, most points of \( N \) have no vector \( F_* \frac{\partial}{\partial t} \), and one point has two such vectors! The fact that r-forms pull back to r-forms is a major advantage!

c) In coordinates, \((u^1, \ldots, u^m)\) on \( M \) and \((x^1, \ldots, x^n)\) on \( N \):

\[
F(u^1, \ldots, u^m) = (x^1(u^1, \ldots, u^m), \ldots, x^n(u^1, \ldots, u^m))
\]

\[
F^* dx^i = \frac{\partial x^i}{\partial u^l} du^l
\]

\[
F^* \left( \sum_{J} a_{J} dx^J \right) = \sum_{J=1}^{n} a_{J} \left( \frac{\partial x^j_1}{\partial u^i_1} du^i_1 \wedge \ldots \wedge \frac{\partial x^j_r}{\partial u^i_r} du^i_r \right)
\]

where \( J = (1 \leq j_1, \ldots, j_r \leq n) \).

d) Example: Let \( \alpha = V_i(x^1, x^2, x^3) dx^i \) be a 1-form on \( \mathbb{R}^3 \) (here \( x^1, x^2, x^3 \) are the standard coordinates on \( \mathbb{R}^3 \)). Let \( F : \mathbb{R} \to \mathbb{R}^3 \) be a curve: \( F(t) = (x^1(t), x^2(t), x^3(t)) \). Then,

\[
F^* \alpha = V_i(x^1(t), x^2(t), x^3(t)) \frac{dx^i}{dt} dt,
\]

a 1-form on \( \mathbb{R} \).

If \( \vec{V} = V_i(x^1, x^2, x^3) \frac{\partial}{\partial x^i} \) is regarded as a force vector field in \( \mathbb{R}^3 \), the work done by \( \vec{V} \) on a particle moving along the curve \( F \), \( t_0 \leq t \leq t_1 \) is

\[
\int_{t_0}^{t_1} \vec{V}_F(t) \cdot \frac{dF}{dt} dt = \int_{t_0}^{t_1} V_i(x^1(t), x^2(t), x^3(t)) \frac{dx^i}{dt} dt
\]

\[
= \int_{t_0}^{t_1} F^* \alpha
\]
• a) Pullback is not a differential operator. It is a linear operator at each point:

\[ F^*(a \alpha + b \beta) = aF^* \alpha + bF^* \beta. \]

and it also satisfies

\[ F^*(\alpha \wedge \beta) = (F^* \alpha) \wedge (F^* \beta) \]

b) Key Fact: For any r-form \( \alpha \), \( F^*(d \alpha) = d(F^* \alpha) \).