Extra credit assignment (for undergraduate students). Due Monday, April 28

For an $n$-manifold $M$ define the $(2n)$-manifold $T^*M$ as

$$T^*M = \cup_{p \in M} M^*_p$$

The set $T^*M$ is called the cotangent bundle of $M$. For every coordinate chart $(U, \phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n)$ on $M$ define a coordinate chart

$$\psi = (x^1, \ldots, x^n, y^1, \ldots, y^n) : T^*U \to \mathbb{R}^{2n}$$

as follows:

$$\psi : \sum_{i=1}^{n} y^i dx^i|_p \mapsto (x^1(p), \ldots, x^n(p), y_1, \ldots, y_n)$$

where $p \in U$, $y_1, \ldots, y_n \in \mathbb{R}$ and $\sum_{i=1}^{n} y_i dx^i|_p \in M^*_p$.

1. For another chart $(U', \phi' = (x'^1, \ldots, x'^n) : U' \to \mathbb{R}^n)$, compute the Jacobian matrix

$$\frac{\partial(x'^1, \ldots, x'^n, y'_1, \ldots, y'_n)}{\partial(x^1, \ldots, x^n, y_1, \ldots, y_n)}$$

on $T^*U \cap T^*U'$.

2. Show that this Jacobian has positive determinant. What does this tell us about the manifold $T^*M$?

3. Show that there exist a 1-form $\theta$ on $T^*M$ given in coordinates by

$$\theta = \sum_{i=1}^{n} y_i dx^i.$$

That is, show that

$$\sum_{i=1}^{n} y_i dx^i = \sum_{i=1}^{n} y'_i d(x'^i)$$

4. Let $\omega = d\theta$ where $\theta$ is as above. Thus $\omega$ is a 2-form on $T^*M$. Show that

$$\omega^n = \omega \wedge \cdots \wedge \omega$$

is $\neq 0$ at every point of $T^*M$. How is this fact related to the conclusion of part (2)?