1. More about the gradient of a function  Let $M$ be an $n$-manifold with a smooth Riemannian metric $g$. Recall that we used $g$ to give, for every $p \in M$, a coordinate-free identification between $M_p$ and $M_p^*$, as follows. To every $\vec{v} \in M_p$ we associated a 1-form $\alpha_{\vec{v}} \in M_p^*$, $\alpha_{\vec{v}} : M_p \to \mathbb{R}$, defined as $\alpha_{\vec{v}}(\vec{w}) := g_p(\vec{v}, \vec{w})$, where $\vec{w} \in M_p$ is arbitrary.

Recall also that for every smooth $f : M \to \mathbb{R}$ we used $g$ to define a vector field $\text{grad} f$ on $M$.

Claim. For every $p \in M$ we have $\alpha_{\text{grad} f|_p} = df|_p$.

Proof. Let $p \in M$ and let $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be a chart on $M$ with $p \in U$.

With respect to this chart,

$$df|_p = \sum_{j=1}^n \frac{\partial f}{\partial x^j}|_p \, dx^j|_p,$$

and

$$\alpha_{\text{grad} f|_p} = \sum_{j=1}^n \alpha_j \, dx^j|_p$$

where $\alpha_j = \alpha_{\text{grad} f|_p}(\frac{\partial}{\partial x^j}|_p)$.

Recall that $\text{grad} f = \sum_i v^i \frac{\partial}{\partial x^i}|_p$, where $v^i = \sum_k g^{ik} \frac{\partial f}{\partial x^k}|_p$, so that

$$\text{grad} f = \sum_i \sum_k g^{ik} \frac{\partial f}{\partial x^k}|_p \frac{\partial}{\partial x^i}|_p$$

To show that $\alpha_{\text{grad} f|_p} = df|_p$, we need to verify that for every $j = 1, \ldots, n$ $\alpha_j = \frac{\partial f}{\partial x^j}|_p$. By definition,

$$\alpha_j = \alpha_{\text{grad} f|_p}(\frac{\partial}{\partial x^j}|_p) = \langle \text{grad} f|_p, \frac{\partial}{\partial x^j}|_p \rangle = \sum_k g^{ik} \frac{\partial f}{\partial x^k}|_p \frac{\partial}{\partial x^j}|_p = \sum_k g^{ik} \frac{\partial f}{\partial x^k}|_p g_{ij} = \sum_k g^{ik} g_{ij} \frac{\partial f}{\partial x^k}|_p = \sum_k g^{ki} \frac{\partial f}{\partial x^i}|_p \frac{\partial}{\partial x^j}|_p$$

since $g^{ik} = g_{ki}$

The equality $\sum_i g^{ki} g_{ij} = \delta^k_j$ used above, comes from the fact that $(g^{ij})_{ij}$ is the inverse of the matrix $(g_{ij})_{ij}$, and so the product of these two matrices is the $n \times n$ identity matrix, with the entry in position $k, j$ being $\sum_i g^{ki} g_{ij} = \delta^k_j$. 

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Thus we have established that $\alpha_j = \frac{\partial f}{\partial x_j} \big|_p$ and therefore $\alpha_{\text{grad} f \mid_p} = df \mid_p$, as claimed.

2. A throw-back to some earlier material:

(1) The tangent map between manifolds satisfies the Chain Rule:

**Theorem. [Chain Rule for maps between manifolds]** Let $M, N, L$ are smooth manifolds, $F : M \to N$, $H : N \to L$ are smooth functions and let $p \in M, q \in N, r \in L$ are such that $F(p) = q$. $H(q) = r$, so that $(H \circ F)(q) = r$. Thus $F_\ast : M_p \to N_q$, $H_\ast : N_q \to L_r$ and $(H \circ F)_\ast : M_p \to L_r$ are linear maps. Then

$$(H \circ F)_\ast = H_\ast \circ F_\ast,$$

that is, for every $\vec{v} \in M_p$

$$(\tilde{\mathbf{1}}) \quad (H \circ F)_\ast(\vec{v}) = H_\ast(F_\ast(\vec{v})).$$

**Proof.** Let $\vec{v} \in M_p$ be arbitrary. Recall that tangent vectors are differential operators on smooth functions. By definition of the tangent map, for every smooth $f : L \to \mathbb{R}$ we have

$$((H \circ F)_\ast \vec{v})(f) = \vec{v}(f \circ H \circ F).$$

Denote $\vec{w} = F_\ast \vec{v} \in N_q$. Then

$$(H_\ast(F_\ast \vec{v}))(f) = (H_\ast \vec{w})(f) = \vec{w}(f \circ H) = (F_\ast \vec{v})(f \circ H) = \vec{v}(f \circ H \circ F).$$

Thus for every smooth $f : M \to \mathbb{R}$ we have

$$((H \circ F)_\ast \vec{v})(f) = (H_\ast(F_\ast \vec{v}))(f)$$

and therefore $(H \circ F)_\ast \vec{v} = H_\ast(F_\ast \vec{v})$. Since this equality holds for every $\vec{v} \in M_p$, it follows that $(H \circ F)_\ast = H_\ast \circ F_\ast$, as required. \hfill $\Box$

**Note:** The interesting feature of this proof is that it is completely coordinate-free and does not directly appeal to the classic Chain Rule from multi-variable calculus. However, the classic Chain Rule is used (and rather heavily) in earlier arguments describing $M_p$ as the set of differential operators of the type $\sum_i v_i \frac{\partial}{\partial x_i} \big|_p$ and in establishing the basic properties of $M_p$.

(2) Let $\gamma : (a, b) \to M$ be a smooth curve. Think about $(a, b)$ as a 1-manifold, with the coordinate $t$.

**Claim.** For every $t_0 \in (a, b)$ we have

$$\dot{\gamma}(t_0) = \gamma'(t_0) \left( \frac{\partial}{\partial t} \big|_{t_0} \right)$$

**Proof.** By definition, $\gamma'(t_0) \in M_{\gamma(t_0)}$ is such that for every smooth $f : M \to \mathbb{R}$

$$\dot{\gamma}(f) = \left. \frac{d}{dt} \right|_{t=t_0} (f \circ \gamma).$$
Similarly, by definition of $\gamma_*$ we have $(\gamma_\ast \vec{v})(f) = \vec{v}(f \circ \gamma)$ for every $\vec{v} \in (a, b)_{t_0}$. Applying this formula to $\vec{v} = \frac{\partial}{\partial t}$ we get 

$$\gamma_*(\frac{\partial}{\partial t}|_{t_0})(f) = \frac{d}{dt}|_{t=t_0}(f \circ \gamma) = (\dot{\gamma})(f).$$

Since the above equality holds for every $f$, it follows that $\dot{\gamma}(t_0) = \gamma_*(\frac{\partial}{\partial t}|_{t_0})$, as claimed.

3. Isometries. Let $M$ be an $n$-manifold with a smooth Riemannian metric $g$. A diffeomorphism $F : M \to M$ is called an isometry with respect to $g$ is $F_*$ preserves $g$, that is, if for every $p \in M$ and all $\vec{v}, \vec{w} \in M_p$ we have 

$$g|_p(\vec{v}, \vec{w}) = g|_{F(p)}(F_*\vec{v}, F_*\vec{w}).$$

By linearity of $F_*$ and multi-linearity of $g$ to verify $\dot{\gamma}$ it is enough to check that $\dot{\gamma}$ holds for all $\vec{v}, \vec{w}$ from some basis of $M_p$, e.g. for all choices of $\vec{v}, \vec{w}$ from $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$, where $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a chart on $M$ with $p \in U$.

Claim. Let $F : M \to M$ be an isometry with respect to $g$. Let $\gamma : [a,b] \to M$ be a smooth curve and let $\beta = F \circ \gamma : [a,b] \to M$.

Then $L(\gamma) = L(\beta)$.

Proof. By definition,

$$L(\gamma) = \int_a^b ||\dot{\gamma}(t)|| \, dt, \quad L(\beta) = \int_a^b ||\dot{\beta}(t)|| \, dt$$

We have

$$\dot{\beta} = \beta_*(\frac{\partial}{\partial t}) = (F \circ \gamma_*)\frac{\partial}{\partial t} = (F_* \gamma_*)(\frac{\partial}{\partial t}) = F_*(\dot{\gamma}).$$

Since $F$ is an isometry, we have

$$||\dot{\gamma}||^2 = g(\dot{\gamma}, \dot{\gamma}) = g(F_*\dot{\gamma}, F_*\dot{\gamma}) = g(\dot{\beta}, \dot{\beta}) = ||\dot{\beta}||^2.$$ 

Hence $||\dot{\gamma}(t)|| = ||\dot{\beta}(t)||$ and therefore, by the above formulas for $L(\gamma)$ and $L(\beta)$, we have $L(\gamma) = L(\beta)$, as claimed.

Example. Consider $M = \{(x, y) \in \mathbb{R}^2 | y > 0\} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ with the coordinates $(x^1, x^2) = (x, y)$ and with the Riemannian metric

$$g_{ij}|_{(x,y)} = \begin{cases} \frac{1}{y^2} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j \in \{1, 2\}$.

The pair $(M, g)$ is called the hyperbolic plane and is often denoted $\mathbb{H}^2$. 

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$. Define a function $F_A : M \to M$ given by $F(z) = \frac{az+b}{cz+d}$ for $z \in M$.

(1) Verify that $F_A$ is indeed a function $M \to M$, that is show that for every $z \in \mathbb{C}$ with $Im(z) > 0$ we have $Im \left( \frac{az+b}{cz+d} \right) > 0$.

(2) Show that $F : M \to M$ is a diffeomorphism. (Hint: Try to find a formula for the inverse map $(F_A)^{-1}$ using the matrix $A^{-1}$.)

(3) Show that $F_A : M \to M$ is an isometry of $M$ with respect to the Riemannian metric $g$ defined above.