Math 481 Exam 2 (SOLUTIONS)

1. Let $W$ be a $(0, 2)$-tensor on a 2-manifold $M$ such that in a chart $(U, \phi = (x^1, x^2))$ we have

\[ W^{1,1} = x^1 + 5, \quad W^{i,j} = 0 \text{ for } (i, j) \neq (1, 1). \]

Let $(V, \psi = (y^1, y^2))$ be another chart such that $V = U$ and such that

\[ y^1 = x^1 + 3x^2, \quad y^2 = -x^1. \]

(a) Find all the coefficients $W^{\prime i,j}$ of $W$ in the chart $(V, \psi)$, as explicit functions of $y^1, y^2$.

(b) In the chart $(U, \phi)$ compute the function

\[ W((x^2)^2 dx^1, 3dx^1 + e^{x_1 x_2} dx^2). \]

Solution.

(a) We have:

\[ W^{\prime i,j} = \sum_{k, \ell} \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^\ell} W^{k,\ell} = \frac{\partial y^i}{\partial x^1} \frac{\partial y^j}{\partial x^1} W^{1,1} = (x^1 + 5) \frac{\partial y^i}{\partial x^1} \frac{\partial y^j}{\partial x^1}. \]

Note that $y^1 = x^1 + 3x^2, \quad y^2 = -x^1$ implies $x^1 = -y^2, x^2 = \frac{1}{3}(y^1 + y^2)$. Hence

\[ W^{1,1} = (x^1 + 5) \cdot 1 \cdot 1 = x^1 + 5 = -y^2 + 5 \]
\[ W^{1,2} = (x^1 + 5) \cdot 1 \cdot -1 = y^2 - 5 \]
\[ W^{2,1} = (x^1 + 5) \cdot -1 \cdot 1 = y^2 - 5 \]
\[ W^{2,2} = (x^1 + 5) \cdot -1 \cdot -1 = -y^2 + 5. \]

(b) We have:

\[ W((x^2)^2 dx^1, 3dx^1 + e^{x_1 x_2} dx^2) = 3(x^2)^2(x^1 + 5). \]

2. Let $\omega$ and $\eta$ be 1-forms on $\mathbb{R}^2$, considered with coordinates $(x, y)$, defined as:

\[ \omega = 2xy dx + (x^2 + 3) dy, \quad \eta = e^{xy} dx - dy. \]

(a) Compute $d\eta$ and $\omega \wedge \eta$.

(b) Consider the vector field $X = x^2 y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Compute the 1-form $i_X(d\eta)$.

Solution.

(a) We have:

\[ d\eta = xe^{xy} dy \wedge dx = -xe^{xy} dx \wedge dy, \]
\[ \omega \wedge \eta = -2xy dx \wedge dy + (x^2 + 3)e^{xy} dy \wedge dx = -(2xy + (x^2 + 3)e^{xy}) dx \wedge dy. \]

(b) We have:
\[ i_X(d\eta) \left( \frac{\partial}{\partial x} \right) = d\eta(X, \frac{\partial}{\partial x}) = d\eta(x^2y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \frac{\partial}{\partial x}) =
\]
\[ = d\eta \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right) = -d\eta \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = x e^{xy}. \]

Similarly:
\[ i_X(d\eta) \left( \frac{\partial}{\partial y} \right) = d\eta(X, \frac{\partial}{\partial y}) = d\eta(x^2y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \frac{\partial}{\partial y}) =
\]
\[ = x^2y d\eta \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = -x^3 ye^{xy}. \]

Therefore
\[ i_X(d\eta) = xe^{xy} dx - x^3 ye^{xy} dy. \]

3.
Let \( \omega \) be a 2-form on \( \mathbb{R}^3 \), with coordinates \((x, y, z)\), given by
\[ \omega = (x z^2 + y) dx \wedge dy. \]

Let \( \sigma = 3 \gamma \) be a 2-chain in \( \mathbb{R}^3 \) where
\[ \gamma(t, s) = (t + s, t, 2), \quad \text{for } (t, s) \in [0, 1]^2. \]

(a) Compute \( \int_\sigma \omega \).
(b) Compute the face-map \( \gamma(2, 0)(t) \), where \( t \in [0, 1] \).
(c) Let \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) (where \( \mathbb{R}^2 \) is taken with coordinates \((u, v)\)) be given by
\[ F(u, v) = (\sin(uv), u^2, v^3). \]

Compute the form \( F^* \omega \).

**Solution.**

(a) We have
\[ \gamma^* \omega = (4(t + s) + t)(dt + ds) \wedge dt = (5t + 4s)ds \wedge dt = -(5t + 4s)dt \wedge ds. \]

Therefore
\[ \int_\gamma \omega = -\int_{[0,1]^2} (5t + 4s) dt ds = -\int_{\gamma}^1 \left( \frac{5}{2} + 4s \right) ds = -\left( \frac{5}{2} + 2 \right) = -\frac{9}{2}. \]

Hence
\[ \int_\sigma \omega = 3 \int_\gamma \omega = -\frac{27}{2}. \]

(b) We have
\[ \gamma(2, 0)(t) = \gamma(t, 0) = (t, t, 2). \]

(c) We have
\[ F^*(dx) = d\sin(uv) = v \cos(uv) du + u \cos(uv) dv \]
and
\[ F^*(dy) = d(u^2) = 2u du. \]
Therefore
\[ F^* \omega = \left(v^6 \sin(uv) + u^2\right) (v \cos(uv) \, du + u \cos(uv) \, dv) \wedge 2u \, du = \]
\[ \left(v^6 \sin(uv) + u^2\right) 2u^2 \cos(uv) \, dv \wedge du = \]
\[ - \left(v^6 \sin(uv) + u^2\right) 2u^2 \cos(uv) \, du \wedge dv. \]

4.
Let \( \omega \) be a smooth \( r \)-form and \( \eta \) be a smooth \( s \)-form on a smooth manifold \( M^n \), where \( r, s \geq 1 \).
(a) Prove that if both \( \omega \) and \( \eta \) are closed then \( \omega \wedge \eta \) is closed.
(b) Prove that if \( \omega \) is exact and \( \eta \) is closed then \( \omega \wedge \eta \) is exact.
(c) Is it always true that \( \omega \wedge \omega = 0 \)? If yes, explain why. If not, provide a counter-example.

**Solution.**
(a) If both \( \omega, \eta \) are closed, then \( d\omega = 0, d\eta = 0 \) and hence
\[ d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta = 0. \]
(b) Suppose that \( \omega \) is exact and \( \eta \) so that \( \omega = d\alpha \) for an \( (r-1) \)-form \( \alpha \) and \( d\eta = 0 \).
Then
\[ d(\alpha \wedge \eta) = d\alpha \wedge \eta + (-1)^{r-1} \alpha \wedge d\eta = d\alpha \wedge \eta + 0 = \omega \wedge \eta. \]
Thus \( \omega \wedge \eta \) is exact.
(c) No, this is not always true. For example, consider \( \mathbb{R}^4 \) with the coordinates \((x, y, z, w)\) and let \( \omega = dx \wedge dy + dz \wedge dw \). Then
\[ \omega \wedge \omega = dx \wedge dy \wedge dz \wedge dw + dz \wedge dw \wedge dx \wedge dy = 2dx \wedge dy \wedge dz \wedge dw \neq 0. \]

**Note.** If \( \omega \) is an \( r \)-form and \( \eta \) is an \( s \)-form, then
\[ \eta \wedge \omega = (-1)^r \omega \wedge \eta \]
and hence
\[ \omega \wedge \omega = (-1)^r \omega \wedge \omega. \]
This if \( r \) is odd, we always have \( \omega \wedge \omega = -\omega \wedge \omega \) and hence \( \omega \wedge \omega = 0 \). But if \( r \) is even, then \( \omega \wedge \omega \) may be different from 0, as the above example illustrates.