Lie Bracket of two vector fields

1. The definition of a Lie bracket.

Let M be an n-manifold. Recall that if X and Y are smooth vector fields on M then X and Y are 1-st order differential operators on smooth functions $M \to \mathbb{R}$. Thus, for a smooth function $f : M \to \mathbb{R}$, Xf and Yf are again smooth functions $M \to \mathbb{R}$.

We now define a differential operator $[X, Y]$, called the *Lie bracket* or the *commutator* of X and Y as

$$[X, Y] := XY - YX,$$

that is, for a smooth function $f : M \to \mathbb{R}$,

$$[X, Y](f) := X(Yf) = Y(Xf).$$

By its definition, $[X, Y]$ is a 2-nd order differential operator. However, by a kind of a miracle, it turns out that $[X, Y]$ is actually a 1-st order differential operator and, in fact, a vector field:

Theorem 1. For any smooth vector fields X and Y on an n-manifold M, the Lie bracket $[X, Y]$ is again a smooth vector field on M.

We will not give a detailed proof of Theorem 1 but instead will only verify that Theorem 1 holds for the basic vector fields.

Let $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ be a coordinate patch on M. Let $X = a(x^1, \dots, x^n)\frac{\partial}{\partial x^i}$ and $Y = b(x^1, \dots, x^n)\frac{\partial}{\partial x^j}$ for some $1 \le i, j \le n$ and for some smooth functions $a = a(x^1, \dots, x^n)$ and $b = b(x^1, \dots, x^n)$.

Let $f : M \to \mathbb{R}$ be a smooth function. We now compute $[X, Y](f)$. By definition:

$$[X, Y](f) = [a\frac{\partial}{\partial x^i}, b\frac{\partial}{\partial x^j}]f = a\frac{\partial}{\partial x^i}(b\frac{\partial f}{\partial x^j}) - b\frac{\partial}{\partial x^j}(a\frac{\partial}{\partial x^i}) =$$

using the product rule

$$a\frac{\partial b}{\partial x^i}\frac{\partial f}{\partial x^j} + ab\frac{\partial^2 f}{\partial x^i \partial x^j} - b\frac{\partial a}{\partial x^j}\frac{\partial f}{\partial x^i} - ba\frac{\partial^2 f}{\partial x^j \partial x^i} =$$

$$a\frac{\partial b}{\partial x^i}\frac{\partial f}{\partial x^j} - b\frac{\partial a}{\partial x^j}\frac{\partial f}{\partial x^i} = (a\frac{\partial b}{\partial x^i}\frac{\partial}{\partial x^j} - b\frac{\partial a}{\partial x^j}\frac{\partial}{\partial x^i})f$$

Therefore

$$(\dagger) \qquad [X, Y] = [a\frac{\partial}{\partial x^i}, b\frac{\partial}{\partial x^j}] = a\frac{\partial b}{\partial x^i}\frac{\partial}{\partial x^j} - b\frac{\partial a}{\partial x^j}\frac{\partial}{\partial x^i}.$$

Note that if $a = b \equiv 1$ then $\frac{\partial b}{\partial x^i} = \frac{\partial a}{\partial x^j} = 0$. Therefore

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0.$$

2. General properties of the Lie bracket

We have:

(1) $[X, Y] = -[Y, X]$.

(2) $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$ and $[X, Y_1 + Y_2] = [X, Y_1] + [X, Y_2]$.

(3) For any smooth functions $a, b : M \to \mathbb{R}$

$$[aX, bY] = ab[X, Y] + a(Xb)Y - b(Ya)X.$$

(4) ["Jacobi Identity"]

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Note that using property (2) above together with formula (†), it is not hard to compute $[\sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x^j}]$ in a coordinate chart $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ on M:

$$[\sum_{i=1}^{n} a^i(x)\frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b^j(x)\frac{\partial}{\partial x^j}] = \sum_i \sum_j [a^i(x)\frac{\partial}{\partial x^i}, b^j(x)\frac{\partial}{\partial x^j}] =$$

$$\sum_i \sum_j a^i \frac{\partial b^j}{\partial x^i}\frac{\partial}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial x^i} \tag{‡}$$

However, as a practical matter, it is better to remember formula (†) together with properties (1),(2),(3),(4) above, rather than to try to memorize formula (‡).

Example. Let $M = \mathbb{R}^3$ with coordinates (x, y, z) and let $X = 2xz\frac{\partial}{\partial x} + e^{yz}\frac{\partial}{\partial y}$ and $Y = (x^2 + y^2 + z^2)\frac{\partial}{\partial x} + 5\frac{\partial}{\partial z}$.

We want to compute $[X, Y]$. We have

$$[X, Y] =$$

$$[2xz\frac{\partial}{\partial x}, (x^2 + y^2 + z^2)\frac{\partial}{\partial x}] + [2xz\frac{\partial}{\partial x}, 5\frac{\partial}{\partial z}] + [e^{yz}\frac{\partial}{\partial y}, (x^2 + y^2 + z^2)\frac{\partial}{\partial x}] + [e^{yz}\frac{\partial}{\partial y}, 5\frac{\partial}{\partial z}].$$

Using (‡), we get

$$[2xz\frac{\partial}{\partial x}, (x^2 + y^2 + z^2)\frac{\partial}{\partial x}] = 2xz2x\frac{\partial}{\partial x} - (x^2 + y^2 + z^2)2z\frac{\partial}{\partial x} =$$

$$(2x^2z - 2y^2z - 2z^3)\frac{\partial}{\partial x},$$

and

$$[2xz\frac{\partial}{\partial x}, 5\frac{\partial}{\partial z}] = 0 \ - 10x\frac{\partial}{\partial x} = -10x\frac{\partial}{\partial x},$$

and

$$[e^{yz}\frac{\partial}{\partial y}, (x^2 + y^2 + z^2)\frac{\partial}{\partial x}] = e^{yz}2y\frac{\partial}{\partial x} - (x^2 + y^2 + z^2) \cdot 0\frac{\partial}{\partial y} = e^{yz}2y\frac{\partial}{\partial x},$$

and

$$[e^{yz}\frac{\partial}{\partial y}, 5\frac{\partial}{\partial z}] = e^{yz} \cdot 0\frac{\partial}{\partial z} - 5ye^{yz}\frac{\partial}{\partial y} = -5ye^{yz}\frac{\partial}{\partial y}.$$

Therefore

$$[X, Y] = \left(2x^2z - 2y^2z - 2z^3 - 10x + 2ye^{yz}\right)\frac{\partial}{\partial x} - 5ye^{yz}\frac{\partial}{\partial y}.$$