Lie Bracket of two vector fields

1. The definition of a Lie bracket.
Let $M$ be an $n$-manifold. Recall that if $X$ and $Y$ are smooth vector fields on $M$ then $X$ and $Y$ are 1-st order differential operators on smooth functions $M \to \mathbb{R}$. Thus, for a smooth function $f : M \to \mathbb{R}$, $Xf$ and $Yf$ are again smooth functions $M \to \mathbb{R}$.

We now define a differential operator $[X,Y]$, called the Lie bracket or the commutator of $X$ and $Y$ as

$$[X,Y] := XY - YX,$$

that is, for a smooth function $f : M \to \mathbb{R}$,

$$[X,Y](f) := X(Yf) = Y(Xf).$$

By its definition, $[X,Y]$ is a 2-nd order differential operator. However, by a kind of a miracle, it turns out that $[X,Y]$ is actually a 1-st order differential operator and, in fact, a vector field:

**Theorem 1.** For any smooth vector fields $X$ and $Y$ on an $n$-manifold $M$, the Lie bracket $[X,Y]$ is again a smooth vector field on $M$.

We will not give a detailed proof of Theorem 1 but instead will only verify that Theorem 1 holds for the basic vector fields.

Let $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be a coordinate patch on $M$. Let $X = a(x^1, \ldots, x^n) \frac{\partial}{\partial x^i}$ and $Y = b(x^1, \ldots, x^n) \frac{\partial}{\partial x^j}$ for some $1 \leq i, j \leq n$ and for some smooth functions $a = a(x^1, \ldots, x^n)$ and $b = b(x^1, \ldots, x^n)$.

Let $f : M \to \mathbb{R}$ be a smooth function. We now compute $[X,Y](f)$. By definition:

$$[X,Y](f) = \left[ a \frac{\partial}{\partial x^i}, b \frac{\partial}{\partial x^j} \right] f = a \frac{\partial}{\partial x^i}(b \frac{\partial f}{\partial x^j}) - b \frac{\partial}{\partial x^j}(a \frac{\partial f}{\partial x^i}) =$$

using the product rule

$$a \frac{\partial b}{\partial x^i} \frac{\partial f}{\partial x^j} + ab \frac{\partial^2 f}{\partial x^i \partial x^j} - b \frac{\partial a}{\partial x^i} \frac{\partial f}{\partial x^j} - ba \frac{\partial^2 f}{\partial x^j \partial x^i} =$$

Therefore

$$[X,Y] = \left[ a \frac{\partial}{\partial x^i}, b \frac{\partial}{\partial x^j} \right] = a \frac{\partial b}{\partial x^i} \frac{\partial}{\partial x^j} - b \frac{\partial a}{\partial x^j} \frac{\partial}{\partial x^i}.$$

Note that if $a = b \equiv 1$ then $\frac{\partial b}{\partial x^i} = \frac{\partial a}{\partial x^j} = 0$. Therefore

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} = 0.$$

2. General properties of the Lie bracket
We have:

(1) $[X,Y] = -[Y,X].$

(2) $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$ and $[X, Y_1 + Y_2] = [X, Y_1] + [X, Y_2].$
(3) For any smooth functions \( a, b : M \to \mathbb{R} \)
\[ [aX, bY] = ab[X, Y] + a(Xb)Y - b(Ya)X. \]

(4) ["Jacobi Identity"]
\[ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \]

Note that using property (2) above together with formula (†), it is not hard to compute\[ \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x^j} \] in a coordinate chart \( \phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n \) on \( M \):
\[
\left[ \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x^j} \right] = \sum_{i} \sum_{j} \left[ a_i(x) \frac{\partial}{\partial x^i}, b_j(x) \frac{\partial}{\partial x^j} \right] = \\
\sum_{i} \sum_{j} a_i \frac{\partial b_j}{\partial x^i} \frac{\partial}{\partial x^j} - b_j \frac{\partial a_i}{\partial x^j} \frac{\partial}{\partial x^i}
\]
(‡)

However, as a practical matter, it is better to remember formula (†) together with properties (1), (2), (3), (4) above, rather than to try to memorize formula (‡).

Example. Let \( M = \mathbb{R}^3 \) with coordinates \( (x, y, z) \) and let \( X = 2xz \frac{\partial}{\partial x} + e^{yz} \frac{\partial}{\partial y} \) and \( Y = (x^2 + y^2 + z^2) \frac{\partial}{\partial x} + 5 \frac{\partial}{\partial z} \).

We want to compute \([X, Y]\). We have
\[ [X, Y] = \]
\[ [2xz \frac{\partial}{\partial x}, (x^2 + y^2 + z^2) \frac{\partial}{\partial x}] + [2xz \frac{\partial}{\partial x}, 5 \frac{\partial}{\partial z}] + [e^{yz} \frac{\partial}{\partial y}, (x^2 + y^2 + z^2) \frac{\partial}{\partial x}] + [e^{yz} \frac{\partial}{\partial y}, 5 \frac{\partial}{\partial z}] . \]

Using (‡), we get
\[ [2xz \frac{\partial}{\partial x}, (x^2 + y^2 + z^2) \frac{\partial}{\partial x}] = 2xz2x \frac{\partial}{\partial x} - 2xz \frac{\partial}{\partial x} + 2y2z \frac{\partial}{\partial x} - (x^2 + y^2 + z^2)2z \frac{\partial}{\partial x} = \\
(2x^2z - 2y^2z - 2z^3) \frac{\partial}{\partial x}, \]
and
\[ [2xz \frac{\partial}{\partial x}, 5 \frac{\partial}{\partial z}] = 0 - 10x \frac{\partial}{\partial x} = -10x \frac{\partial}{\partial x}, \]
and
\[ [e^{yz} \frac{\partial}{\partial y}, (x^2 + y^2 + z^2) \frac{\partial}{\partial x}] = e^{yz}2y \frac{\partial}{\partial x} - (x^2 + y^2 + z^2) \frac{\partial}{\partial x} - e^{yz}2y \frac{\partial}{\partial x}, \]
and
\[ [e^{yz} \frac{\partial}{\partial y}, 5 \frac{\partial}{\partial z}] = e^{yz} \cdot 0 \frac{\partial}{\partial z} - 5ye^{yz} \frac{\partial}{\partial y} = -5ye^{yz} \frac{\partial}{\partial y} , \]

Therefore
\[ [X, Y] = (2x^2z - 2y^2z - 2z^3 - 10x + 2ye^{yz}) \frac{\partial}{\partial x} - 5ye^{yz} \frac{\partial}{\partial y}. \]