Manifolds-with-boundary

1. Definition and examples.

For $n \geq 1$ we denote $\mathbb{R}_+^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$. A subset $U$ of $\mathbb{R}_+^n$ is said to be open in $\mathbb{R}_+^n$ if $U = \mathbb{R}_+^n \cap U'$ for some open subset $U'$ of $\mathbb{R}^n$.

A smooth $n$-manifold-with-boundary $M$ is defined in a similar way to the notion of an $n$-manifold, except that the definition of an atlas $\mathcal{A}$ on $M$ is modified as follows. A chart in $\mathcal{A}$ is now allowed to be either a 1-to-1 map $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ such that $\phi(U)$ is open in $\mathbb{R}^n$ (such charts are called regular charts or a 1-to-1 map $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}_+^n$ such that $\phi(U)$ is open in $\mathbb{R}_+^n$ (such charts are called boundary charts). All the other requirements for being an atlas remain the same.

Let $M$ be an $n$-manifold-with-boundary given by an atlas $\mathcal{A}$.

A point $p \in M$ is called a regular point if there exists a chart $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ in $\mathcal{A}$ such that $\phi(U)$ is open in $\mathbb{R}^n$ and such that $p \in U$.

A point $p \in M$ is called a boundary point if $p$ is not a regular point. This implies that whenever $p \in U$, where $U$ is the domain of some chart $\phi$, then $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}_+^n$ is a boundary chart with $p \in U$ such that $x^n(p) = 0$.

The set of all boundary points of $M$ is denoted $\partial M$ and the set of all regular points of $M$ is denoted $int(M)$.

Examples.

1. If $M$ is an $n$-manifold (in the old sense) then $M$ is an $n$-manifold-with-boundary with $int(M) = M$ and $\partial M = \emptyset$.
2. $M = [0, 1]$ is a 1-manifold-with-boundary with $int(M) = (0, 1)$ and $\partial M = \{0, 1\}$.
3. In $n \geq 1$ and $M = \mathbb{R}_+^n$ then $M$ is an $n$-manifold-with-boundary with $int(M) = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$ and $\partial M = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.
4. The closed $n$-ball $D_n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid (x^1)^2 + \cdots + (x^n)^2 \leq 1\}$ is an $n$-manifold-with-boundary with $int(D_n)M = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid (x^1)^2 + \cdots + (x^n)^2 < 1\}$ and with $\partial D_n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid (x^1)^2 + \cdots + (x^n)^2 = 1\} = S^{n-1}$.
5. The “solid torus” $M = D_2 \times S^1$ is a 3-manifold-with-boundary with $\partial M = S^1 \times S^1$ being the 2-torus.

The notions of the tangent space, tangent map, tensors and tensor fields, orientability, differential forms etc are defined for manifolds-with-boundary in exactly the same way as for ordinary manifolds. Note, however, that if $M$ is an $n$-manifold-with-boundary, $p \in \partial M$ and $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a boundary chart with $p \in U$ and $x^n(p) = 0$, we still have $M_p = \{(\sum_{i=1}^n v_i \frac{\partial}{\partial x^i}) \mid v^i \in \mathbb{R}\}$. Thus in this case $M_p$ is still an $n$-dimensional vector space and vectors $\sum_{i=1}^n v_i \frac{\partial}{\partial x^i}$ with $v^n < 0$ are still considered to be elements of $M_p$ (even though they point “away” from $int(M)$).
2. Basic facts

(1) If $M$ is an $n$-manifold-with-boundary, then $\partial M$ is an $(n-1)$-manifold (in the old sense) and $\text{int}(M)$ is an $n$-manifold (again in the old sense). In particular, $\partial \partial M = \emptyset$.

(2) Let $M$ be an $n$-manifold-with-boundary. If $p \in \text{int}(M)$ then $M_p = (\text{int} M)_p$. If $p \in M$ and $p \in \partial M$ and $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a boundary chart with $p \in U$ and $x^n(p) = 0$, then $(\partial M)_p = \{ \sum_{i=1}^n v_i \frac{\partial}{\partial x^i} | v_i \in \mathbb{R}, v^n = 0 \}$.

(3) Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $N = \{ x \in \mathbb{R}^n | F(x) = 0 \}$ is nonempty and such that for every $x \in N$ we have $\text{grad} F|_x \neq (0, \ldots, 0)$. Then $M = \{ x \in \mathbb{R}^n | F(x) \leq 0 \}$ is an $n$-manifold-with-boundary and $\partial M = N$.

(4) If $M$ is an $n$-manifold-with-boundary which is compact, then $\partial M$ is also compact.

(5) If $M$ is an $n$-manifold-with-boundary which is orientable then $\partial M$ is also orientable. If $M$ has an orientation (given by an orientation on each $M_p$), then the induced orientation on $\partial M$ is defined as follows. Let $p \in \partial M$ and let $\vec{v}_1, \ldots, \vec{v}_{n-1}$ be a basis of $(\partial M)_p$. We say that this basis is positively oriented if the basis $\vec{v}, \vec{v}_1, \ldots, \vec{v}_{n-1}$ of $M_p$ is positively oriented for $M_p$, where $\vec{v} \in M_p$ is any vector pointing “away” from $\text{int}(M)$ (such as $\vec{v} = -\frac{\partial}{\partial x^n}|_p$ for a boundary chart $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a boundary chart with $p \in U$ and $x^n(p) = 0$).