Riemannian Connections

1. General Definition of a Riemannian Connection.
   Let \((M, g)\) be a smooth manifold with a smooth Riemannian metric \(g\). Let \(\mathcal{V}\) be the set of all smooth vector fields on \(M\).
   A connection or a covariant derivative on \(M\) is an operator \(\nabla : \mathcal{V} \times \mathcal{V} \to \mathcal{V}\), that, using the notation \(\nabla(X,Y) = \nabla_X Y\), satisfies the following axioms:
   1. \(\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y\)
   2. \(\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2\)
   3. \(\nabla_{fX} Y = f \nabla_X Y\), where \(f : M \to \mathbb{R}\) is any smooth function.
   4. \(\nabla_X (f Y) = X(f) Y + f \nabla_X Y\), where \(f : M \to \mathbb{R}\) is any smooth function.

   A connection \(\nabla\) on \(M\) is called a Riemannian connection with respect to \(g\) if in addition to the above axioms, \(\nabla\) also satisfies the following two axioms:
   5. \(X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\) for any \(X, Y, Z \in \mathcal{V}\).
   6. \(\nabla_X Y - \nabla_Y X = [X, Y]\).

   **Theorem.** For any smooth manifold \(M\) with a smooth Riemannian metric \(g\) there exists a unique Riemannian connection \(\nabla\) on \(M\) corresponding to \(g\).

2. The case of \(\mathbb{R}^n\).
   Let \(M = \mathbb{R}^n\) with the standard inner product giving on \(\mathbb{R}^n\). Then \(\nabla_X Y := D_X Y\) is a Riemannian connection on \(M\). Here \(D_X Y\) is the directional derivative of \(Y\) with respect to \(X\):
   \[
   D_X Y|_p := \lim_{t \to 0} \frac{Y(p + tX) - Y(p)}{t} = \lim_{t \to 0} \frac{Y(c(t)) - Y(p)}{t}
   \]
   where \(c(t)\) is any smooth curve with \(c(0) = p\) and \(c'(0) = X(p)\).

   Let \(c(t)\) be a curve and \(Y(t)\) be a vector field along \(c\), that is for every \(t\) \(Y(t)\) is a tangent vector to \(\mathbb{R}^3\) at the point \(c(t)\). Then the directional derivative of \(Y\) with respect to \(c\) is:
   \[
   D_c Y(t) = \lim_{s \to 0} \frac{Y(t + s) - Y(t)}{s}
   \]
   Note that \(D_c \dot{c} = \ddot{c}\).

3. Riemannian connection on a surface in \(\mathbb{R}^3\).
Let $M^2 \subseteq \mathbb{R}^3$ be an oriented surface with an outward unit normal $\mathbf{n}$. We endow $M$ with the Riemannian metric $g$ obtained by restricting the standard inner product in $\mathbb{R}^3$ to the tangent vectors to $M$.

For any tangent fields $X, Y$ to $M$ in $\mathbb{R}^3$ define

$$\nabla_X Y := D_X Y - \langle D_X Y, \mathbf{n} \rangle \mathbf{n},$$

where $D_X Y$ is the directional derivative of $Y$ with respect to $X$. Then $\nabla$ is a Riemannian connection on $M$ [Check that all the axioms hold!]. Note that in the above definition $\nabla_X Y$ is the tangential (with respect to $M$) component of the vector $D_X Y$. That is, $\nabla_X Y$ is obtained by taking the orthogonal projection of the vector $D_X Y$ to the tangent space of $M$ at the given point.

In view of the uniqueness of Riemannian connections $\nabla$ is in fact the only Riemannian connection on $M$.

4. **Riemannian connection in a chart.**

Let $(M^n, g)$ be a smooth manifold with a smooth Riemannian metric $g$. Let $\nabla$ be the unique Riemannian connection on $M$ corresponding to $g$. Let $(U, \phi = (x^1, \ldots, x^n))$ be a chart on $M$.

We will use the notation $\nabla_i Y := \nabla_{\frac{\partial}{\partial x^i}} Y$.

(1) The Christoffel symbols $\Gamma^k_{ij}$ and $\Gamma_{ij,k}$ are defined as follows:

$$\nabla_i \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

and

$$\Gamma_{ij,k} = g(\nabla_i \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}).$$

(2) The Christoffel symbols can be computed as follows:

$$\Gamma_{ij,k} = \frac{1}{2} \left( - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} \right),$$

and

$$\Gamma^k_{ij} = \sum_m g^{km} \Gamma_{ij,m}.$$

(3) We always have $\Gamma_{ij,k} = \Gamma_{ji,k}$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$ since $\nabla_i \frac{\partial}{\partial x^j} = \nabla_j \frac{\partial}{\partial x^i}$ in view of Axiom (6) of covariant derivative.

(4) Let $X = \sum a^i \frac{\partial}{\partial x^i}$ and $Y = \sum b^j \frac{\partial}{\partial x^j}$. Then
\[ \nabla_X Y = \sum_k \left[ \sum_i a^i \frac{\partial b^k}{\partial x^i} + \sum_{i,j} a^i b^j \Gamma^k_{ij} \right] \frac{\partial}{\partial x^k}. \]

Note that \((\nabla_X Y)|_p\) depends only on \(X(p)\) and on the restriction of \(Y\) to some neighborhood of \(p\) on \(U\) (or even to some curve through \(p\)).

5. Covariant derivative for a vector field along a curve.

Let \(c(t)\) be a smooth regular curve in \(M\) (here \textit{regular} means that \(\dot{c}(t) \neq 0\) for every \(t\)). A \textit{vector field} \(Y\) along \(c\) is a function \(Y(t)\) such that for every \(t\) we have \(Y(t) \in M_{c(t)}\).

Let \(\nabla\) be a Riemannian connection on \(M\) corresponding to a Riemannian metric \(g\). Then \(\nabla_c Y\) is a vector field along \(c\) defined as follows. Let \((U, \phi = (x^1, \ldots, x^n))\) be a chart. In this chart \(c(t)\) has the form \((c^1(t), \ldots, c^n(t))\) where \(c^i(t) = x^i(c(t))\), and \(Y\) has the form

\[ Y(t) = \sum_k b^k(t) \frac{\partial}{\partial x^i}|_{c(t)}. \]

Then

\[ \nabla_c Y(t) = \sum_k \left[ \dot{b}^k(t) + \sum_{i,j} \dot{c}^i(t) b^j(t) \Gamma^k_{ij}(c(t)) \right] \frac{\partial}{\partial x^k}|_{c(t)}. \]

Note that this definition is chosen in such a way that if \(\tilde{Y}\) is a vector field on \(M\) such that for every \(t\) we have \(Y(t) = \tilde{Y}|_{c(t)}\) then for every \(t\) we have

\[ \nabla_c Y(t) = \nabla_{\dot{c}(t)} \tilde{Y}. \]

6. Parallel vector fields and geodesics.

Let \((M^n, g)\) be a smooth manifold with a smooth Riemannian metric \(g\). Let \(\nabla\) be the unique Riemannian connection on \(M\) corresponding to \(g\).

Let \(c(t)\) be a smooth regular curve in \(M\) and let \(Y(t)\) be a vector field along \(c\). We say that \(Y\) is \textit{parallel along} \(c\) if \(\nabla_c Y = 0\).

We say that a smooth regular curve \(c\) is a \textit{geodesic} in \((M, g)\) if \(\dot{c}\) is parallel along \(c\), that is, if \(\nabla_c \dot{c} = 0\).

(1) Let \(M^2 \subseteq \mathbb{R}^3\) be an oriented surface in \(\mathbb{R}^3\) as in Section 3 and let \(\nabla\) be a Riemannian connection on \(M\) given by the explicit formula from Section 3. Recall that \(D_c \dot{c} = \ddot{c}\). Then the condition \(\nabla_c \dot{c} = 0\) is equivalent to

\[ 0 = \ddot{c} - \langle \ddot{c}, n \rangle n, \]

that is, that \(\ddot{c}\) is parallel to \(n\) and thus perpendicular to \(M_{c(t)}\).
(2) If \( Y, Z \) are parallel along a regular curve \( c \) then \( g(Y(t), Z(t)) = const. \) In particular \( ||Y(t)|| = g(Y, Y)^{1/2} = const. \) Thus if \( c(t) \) is a geodesic, then \( ||\dot{c}(t)|| = const. \)

(3) Let \((U, \phi = (x^1, \ldots, x^n))\) be a chart on \( M. \) Let

\[
Y(t) = \sum_k b^k(t) \frac{\partial}{\partial x^i}|c(t),
\]

and let \( c^i(t) = x^i(c(t)). \) Then \( Y \) is parallel along \( c \) if and only if

\[
\dot{b}^k(t) + \sum_{i,j} \dot{c}^i(t) \dot{b}^j(t) \Gamma^k_{ij}(c(t)) = 0, \quad k = 1, \ldots, n.
\]

(4) Let \( c(t) \) be a smooth regular curve as in (1) above. Then \( c \) is a geodesic if and only if it satisfies:

\[
\ddot{c}^k(t) + \sum_{i,j} \dot{c}^i(t) \dot{c}^j(t) \Gamma^k_{ij}(c(t)) = 0, \quad k = 1, \ldots, n.
\]

7. Facts about geodesics.

Let \((M, g)\) be a smooth manifold with a smooth Riemannian metric \( g \) and let \( p \in M. \)

(1) For any \( v \in M_p, v \neq 0 \) there is some \( \epsilon > 0 \) such that there exists a unique geodesic \( c : (-\epsilon, \epsilon) \to M \) with \( c(0) = p \) and \( \dot{c}(0) = v. \)

(2) Let \( c_1 : [0, a_1] \to M \) and \( c_2 : [0, a_2] \to M \) be geodesics such that \( c_1(0) = c_2(0) = p \) and \( \dot{c}_1(0) = \dot{c}_2(0). \) Then \( c_1(t) = c_2(t) \) for every \( t \in [0, \min\{a_1, a_2\}]. \)

(3) If \( M \) is compact then for any \( v \in M_p, v \neq 0 \) there exists a unique geodesic \( c : (-\infty, \infty) \to M \) with \( c(0) = p \) and \( \dot{c}(0) = v. \)

(4) There is some connected open set \( U \) in \( M \) containing \( p \) with the following properties:

(a) For every \( q \in U, q \neq p \) there exists a unique unit speed geodesic \( c_{p,q} \) connecting \( p \) to \( q. \)

(b) For any other curve \( \gamma \) from \( p \) to \( q \) we have \( L(c_{p,q}) \leq L(\gamma). \) Moreover, if \( L(c_{p,q}) = L(\gamma) \) and \( \gamma \) is a unit speed curve then \( \gamma = c_{p,q}. \)

(5) Let \( c : [0, a] \to M \) be a regular smooth curve with \( c(0) = p. \) Then for any \( v \in M_p \) there exists a unique vector field \( Y(t) \) parallel along \( c \) such that \( Y(0) = v. \) In this case one says that \( w = Y(a) \) is obtained from \( v \) by a parallel transport along \( c. \)