The volume form.

1. Orientation on vector spaces and manifolds.

(1) Let $V$ be a vector space of dimension $n \geq 1$. Recall that an orientation on $V$ is specified by choosing a particular basis $e_1, \ldots, e_n$ of $V$. If $v_1, \ldots, v_n$ is any other basis of $V$, this basis is declared positive if the transition matrix $A$ has $\det(A) > 0$ and negative if $\det(A) < 0$.

Here $A = (v^i_j)_{ij}$, where $v_i = \sum_{j=1}^n v^i_j e_j$. Note that if $v_1 = e_1, \ldots, v_n = e_n$, then $A = I_n$ is the identity matrix and hence $e_1, \ldots, e_n$ is a positive basis.

(2) Let $M^n$ be an $n$-manifold. An orientation of $M$ can be viewed as a continuous (with respect to varying $p$) choice of orientations on all the tangent spaces $M_p$, where $p \in M$.

Let $(U_i, \phi_i)_i$ be an orienting atlas on $M$ (that is the Jacobians of all the transition maps $\phi_i \circ \phi_j^{-1}$ have positive determinants). For $p \in M$ this atlas defines an orientation on $M_p$ by declaring that the basis $\frac{\partial}{\partial x^i} |_p, \ldots, \frac{\partial}{\partial x^n} |_p$ is a positive basis of $M_p$, where $p \in (U_i, \phi_i = (x^1, \ldots, x^n))$.

Conversely, suppose that we have specified a a continuous (with respect to varying $p$) choice of orientations on all the tangent spaces $M_p$, where $p \in M$.

We can construct an orienting atlas on $M$ as follows. Start with any atlas $(U_i, \phi_i = (x^1_i, \ldots, x^n_i))_i$ with connected sets $U_i$. For every $U_i$ choose $p \in U_i$ and check whether $\frac{\partial}{\partial x^i} |_p, \ldots, \frac{\partial}{\partial x^n} |_p$ is a positive basis of $M_p$. If yes, keep $(U_i, \phi_i)$ without change. If not, interchange $x^i$ and $x^j$ in $\phi_i$. Do this for every $i$. The resulting collection of charts is an orienting atlas on $M$.

2. The volume form on a vector space. Let $V$ be a vector space of dimension $n \geq 1$ with a chosen orientation. Let $\langle \ , \ \rangle$ be a positive-definite inner product on $V$. The volume form on $V$ corresponding to this choice of an orientation and to $\langle \ , \ \rangle$ is the unique $n$-form $\omega$ on $V$ such that for any positive basis $v_1, \ldots, v_n$ of $V$ we have

$$\omega = \sqrt{|g|} v^1 \wedge \cdots \wedge v^n,$$

here $|g|$ is the determinant of the $n \times n$ symmetric matrix $g$ with $g_{ij} = \langle v_i, v_j \rangle$.

**Facts.**

(1) The form defined by (†) does not depend on the choice of a positive basis $v_1, \ldots, v_n$ of $V$.

(2) If $e_1, \ldots, e_n$ is a positive orthonormal basis of $V$ then $\omega = e^1 \wedge \cdots \wedge e^n$.

(To see this, just apply the formula (†) in this case.

(3) Let $e_1, \ldots, e_n$ be a positive orthonormal basis of $V$. Let $v_1, \ldots, v_n$ be any other basis of $V$ with $v_i = \sum_{j=1}^n v^i_j e_j$. Let $A$ be the transition matrix $A = (v^i_j)_{ij}$.

Then

$$\omega(v_1, \ldots, v_n) = |\det(A)|.$$

Thus $\omega(v_1, \ldots, v_n)$ is equal to the volume of the $n$-dimensional box with sides $v_1, \ldots, v_n$ in $V$, computed using the standard formulas of Euclidean geometry.
(4) Let \( \text{dim}(V) = 1 \). Let \( v \in V, v \neq 0 \), so that \( \{v\} \) is a basis of \( V \). Then \( g \) is a \( 1 \times 1 \)-matrix with the entry \( g_{1,1} = \langle v, v \rangle = ||v||^2 \). Hence \( \omega = ||v||v^* \) if \( v \) is positive (in the sense of the orientation on \( V \)) and \( \omega = -||v||v^* \) if \( v \) is negative. In particular if \( \epsilon \in V \) is a positive unit vector, then \( \omega = \epsilon^* \). Note that if \( w \in V \) is any other vector, then \( w = ce \), where \( c = \epsilon ||w|| \), where \( \epsilon = 1 \) if \( w \neq 0 \) is a positive vector, \( \epsilon = -1 \) if \( w \neq 0 \) is negative and \( \epsilon = 0 \) if \( w = 0 \). Then \( \omega(w) = e^*(ce) = c = \epsilon ||w|| \). Thus the volume form \( \omega(w) \) computes the “signed length” of a \( w \) in this case.

3. The volume form on a manifold. Let \( M^n \) be an oriented \( n \)-manifold with a Riemannian metric \( g \). The volume form \( d\text{vol}_g \) is an \( n \)-form on \( M \) such that if \( (U, \phi = (x^1, \ldots, x^n)) \) is a chart from an orienting atlas then in this chart

\[
d\text{vol}_g = \sqrt{|g|} \, dx^1 \wedge \ldots \wedge dx^n
\]

where \( g \) is the \( n \times n \)-matrix with \( g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \).

4. The volume form on a surface. Let \( M^2 \) be an oriented surface with a Riemannian metric \( g \) and with an orienting chart \( (U, \phi = (x^1, x^2)) \).

Then in this chart

\[
dA = d\text{vol}_g = \sqrt{EF-G^2} \, dx^1 \wedge dx^2,
\]

where \( E = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}), F = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}) \) and \( G = g(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}) \).

5. Outward unit normals for a surface in \( \mathbb{R}^3 \). Let \( M^2 \) be a surface in \( \mathbb{R}^3 \), where \( \mathbb{R}^3 \) is considered with coordinates \((x, y, z)\).

Here for \( p \in M \) we identify the tangent space \( M_p \) with the set of geometric tangent vectors to \( M \) at \( p \) in \( \mathbb{R}^3 \), so that \( M_p \subseteq \mathbb{R}^3 \) is a 2-dimensional linear subspace.

Recall that if \((U, \phi = (x^1, x^2))\) is a chart on \( M \) and \( \psi = \phi^{-1} : U \to \mathbb{R}^3, \psi = (\psi_1 = x(x^1, x^2), \psi_2 = y(x^1, x^2), \psi_3 = z(x^1, x^2)) \) then an abstract tangent vector \( \frac{\partial}{\partial x^1}|_p + b \frac{\partial}{\partial x^2}|_p \in M_p \) is identified with the geometric tangent vector

\[
\left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right)|_p \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \frac{\partial}{\partial x^1}|_p + b \frac{\partial}{\partial x^2}|_p \\ a \frac{\partial}{\partial x^1}|_p + b \frac{\partial}{\partial x^2}|_p \end{bmatrix} \in \mathbb{R}^3
\]

for any \( a, b \in \mathbb{R} \).

(1) If \( M^2 \) is an oriented surface, then at every point in \( p \in M \) one can define the outward unit normal \( n(p) \in \mathbb{R}^3 \). Note that there are precisely two (opposite) vectors of length 1 in \( \mathbb{R}^3 \) that are perpendicular to the 2-dimensional subspace \( M_p \subseteq \mathbb{R}^3 \). The outward unit normal \( n(p) \) is the unique unit normal to \( M_p \) with the property that if \( v_1, v_2 \in M_p \subseteq \mathbb{R}^3 \) is a positive basis of \( M_p \) then \( n(p), v_1, v_2 \) is a positive basis of \( \mathbb{R}^3 \) with respect to the standard orientation on \( \mathbb{R}^3 \).

Specifically, in this case

\[
n(p) = \frac{v_1 \times v_2}{||v_1 \times v_2||}
\]
(2) Suppose $M^2$ is an oriented surface and $(U, \phi = (x^1, x^2))$ is an orienting chart with $\psi = \phi^{-1} = (\psi_1 = x(x^1, x^2), \psi_2 = y(x^1, x^2), \psi_3 = z(x^1, x^2))$.

Then for $p \in U$ the geometric tangent corresponding to $\frac{\partial}{\partial x^1}|_p$ is $\frac{\partial \psi_1}{\partial x^1}|_p \in \mathbb{R}^3$ and the geometric tangent corresponding to $\frac{\partial}{\partial x^2}|_p$ is $\frac{\partial \psi_2}{\partial x^2}|_p \in \mathbb{R}^3$. Hence we can compute the outward unit normal as:

$$n(p) = \frac{\frac{\partial \psi_1}{\partial x^1}|_p \times \frac{\partial \psi_2}{\partial x^2}|_p}{||\frac{\partial \psi_1}{\partial x^1}|_p \times \frac{\partial \psi_2}{\partial x^2}|_p||}$$

(3) Let $f : \mathbb{R}^3$ is a smooth function such that for every $p \in M$ with

$$M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}$$

we have $\text{grad } f_p \neq 0$ (so that $M$ is a 2-manifold).

Let

$$N = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) \leq 0\}.$$ 

Then $N$ is a 3-manifold-with-boundary in $\mathbb{R}^3$ and $\partial N = M$. We give $N$ the same orientation as in $\mathbb{R}^3$ and we give $M = \partial N$ the induced orientation. Then for any $p \in M$ we can compute the outward unit normal as

$$n(p) = \frac{\text{grad } f|_p}{||\text{grad } f|_p||}.$$ 

(4) Let $M^2 \subseteq \mathbb{R}^3$ be a surface. Suppose that for every $p \in M$ we have chosen a unit vector $n(p) \in \mathbb{R}^3$ that is perpendicular to $M_p$ and such that $n(p)$ varies continuously with $p$. This defines an orientation on $M$. Namely, for $p \in M$ and a basis $v_1, v_2, \in M_p$ we declare that $v_1, v_2$ is a positive basis of $M_p$ if and only if $n(p), v_1, v_2$ is a positive basis of $\mathbb{R}^3$ (that is, if and only if $n(p) = cv_1 \times v_2$ where $c > 0$). This determines an orientation on $M_p$ with respect to which $n(p)$ is the outward unit normal.

6. **The volume form for a surface in $\mathbb{R}^3$.** Let $M^2 \subseteq \mathbb{R}^3$ be an oriented surface with an outward unit normal $n = (n^1, n^2, n^3)$. We endow $M$ with a Riemannian metric $g$ that is the restriction to $M_p$ of the standard inner product in $\mathbb{R}^3$. That is, if $v_1 = (v_1^1, v_1^2, v_1^3)$ and $v_2 = (v_2^1, v_2^2, v_2^3)$ are elements of $M_p \leq \mathbb{R}^3$, we put

$$g|_p(v_1, v_2) = (v_1, v_2) = v_1^1v_2^1 + v_1^2v_2^2 + v_1^3v_2^3.$$ 

Then the volume form $dA$ on $M$ is:

$$dA = n^1\,dy \wedge dz + n^2\,dz \wedge dx + n^3\,dx \wedge dy.$$ 

Moreover,

$$n^1\,da = dy \wedge dz, \quad n^2\,da = dz \wedge dx, \quad n^3\,da = dx \wedge dy$$ 

on $M$.

7. **The volume form for a curve in $\mathbb{R}^3$.** Let $M^1 \subseteq \mathbb{R}^3$ be an oriented 1-manifold (that is, a curve in $\mathbb{R}^3$ with a chosen direction). For every point $p \in M$ there are exactly two (opposite) unit tangent vectors to $M$ at $p$. Let $T|_p = (T_1|_p, T_2|_p, T_3|_p) \in \mathbb{R}^3$ be the unique unit tangent vector to $M$ at $p$
which is positively oriented with respect to the orientation on $M$. We endow $M$ with a Riemannian metric $g$ that is the restriction to $M_p$ of the standard inner product in $\mathbb{R}^3$. Denote by $ds$ the volume form on $M$ corresponding to $g$. Then

$$ds = T^1 dx + T^2 dy + T^3 dz$$

on $M$. Moreover,

$$T^1 ds = dx, \ T^2 ds = dy, \ T^3 ds = dz$$

on $M$. 