1. Let $M$ be a 2-manifold and let $W$ be a $(1, 1)$-tensor field on $M$. Let $(U, \Psi = (x^1, x^2))$ and $(V, \Psi = (y^1, y^2))$ be two coordinate charts on $M$, such that in the first chart $W$ has the form

$$W = (x^1)^2 x^2 \left( \frac{\partial}{\partial x^1} \otimes dx^2 \right),$$

that is

$$W^1 = (x^1)^2 x^2 \quad \text{and} \quad W_1 = 0.$$  

a) Express $W$ on $U \cup V$ with respect to the chart $(V, \Psi)$.
b) Suppose that on $U \cap V$ the transition map from $\psi$ to $\psi'$ is $y'_1 = 2x'_1$, $y'_2 = x'_1x'_2$. Find explicitly, as functions of $y'_1, y'_2$, the coefficients $(W')^i_j$ of $W$ with respect to $(V, \psi)$ on $U \cap V$. 
c) Let \( p \in M \) be such that \( p \in U \cap V \) and that \( \Psi(p) = (5, 1) \). Assuming the transition map as in b) find \( W(p) \) \( \left( dy^1 + dy^2, \frac{2}{dy^1} \right) \).
2. Let $M^n$ be a manifold with a smooth Riemannian metric $g$. We say that a diffeomorphism $\alpha : M \rightarrow M$ is an isometry of $(M, g)$ if for every $p \in M$ and $\vec{v}, \vec{w} \in T_p M$ we have
\[ g(p) (\nabla, \nabla) = g(\alpha(p)) (\alpha_* \vec{v}, \alpha_* \vec{w}) \]

a) Suppose $\alpha : M \rightarrow M$ is an isometry of $(M, g)$. Let $\gamma : [0, 1] \rightarrow M$ and $\gamma_1 = \alpha \circ \gamma$.
Prove that the curves $\gamma, \gamma_1$ have equal lengths with respect to $g$.

Hint: You do not need to use coordinate-wise formulas here. Use the definition of length, the definition of $\gamma(t)$ and the fact that for smooth maps $f_1$ and $f_2$ we have $(f_2 \circ f_1)_* = (f_2)_* \circ (f_1)_*$.
Let $M = \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$ and let
\[
G = \frac{(dx)^2 + (dy)^2}{y^2} = \frac{1}{y^2} (dx \otimes dx) + \frac{1}{y^2} (dy \otimes dy)
\] on $M$. Prove that for every $\lambda > 0$ the map
\[
\alpha : M \to M \quad \alpha(x, y) = (\lambda x, \lambda y)
\] is an isometry of $(M, G)$. 
c) For \((M, g)\) above let
\[ \gamma : [0, T] \rightarrow M \text{ be defined as} \]
\[ \gamma(t) = (0, 1+t), \quad t \in [0, T] \quad (T > 0) \]
Find the length of \(\gamma\) with respect to \(g\).

d) Consider the vector field \(V = \frac{x^2}{y} \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial x}\) on \(M\).
Let \(\omega\) be a 1-form on \(M\) defined as
\[ \omega(p) \left( \overrightarrow{w} \right) = g(p) \left< V(p), \overrightarrow{w} \right> \quad \text{for every } p \in M, \overrightarrow{w} \in T_p M. \]
Find explicit representation of \(\omega\) as
\[ \omega = h_1(x, y) \, dx + h_2(x, y) \, dy \quad \text{on } M. \]