M = \{(x',x^2)\} \text{ let } y' = (x')^2 + (x^2)^2, y^2 = x'x^2.

@ For what points (x', x^2) can you NOT use (y', y^2) as coordinates on some open set around (x', x^2)? Explain.

\[
\begin{pmatrix}
\frac{\partial(y', y^2)}{\partial(x', x^2)}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial y'}{\partial x'} & \frac{\partial y'}{\partial x^2} \\
\frac{\partial y^2}{\partial x'} & \frac{\partial y^2}{\partial x^2}
\end{pmatrix} =
\begin{pmatrix}
2x' & 2x^2 \\ x^2 & x'
\end{pmatrix}
\] So (y', y^2) will NOT work as coordinates around a point on these lines.

\[
det = 2(x')^2 - (x^2)^2 = 0 \text{ on lines } y^2 = \pm x' \text{ (see } \mathbb{R}, \text{ Lecture B)}.
\]

6. Let p_0 be the point with (x'_0, x^2_0) = (1, \frac{1}{2}). Express \( \frac{\partial y'}{\partial p_0} \text{ and } \frac{\partial y^2}{\partial p_0} \) as linear combinations of \( \frac{\partial y'}{\partial x'} |_{p_0} \text{ and } \frac{\partial y^2}{\partial x^2} |_{p_0} \).

\[
\begin{pmatrix}
\frac{\partial y'}{\partial x'} & \frac{\partial y^2}{\partial x^2}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial y'}{\partial x'} & \frac{\partial y^2}{\partial x^2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x'_0}{\partial x'} & \frac{\partial x^2_0}{\partial x^2}
\end{pmatrix}^{-1}
\]

at p_0 = (1, \frac{1}{2}),

\[
\left(\frac{\partial y'}{\partial x'} & \frac{\partial y^2}{\partial x^2}\right)
\begin{pmatrix}
\frac{2}{3} & 1 \\
\frac{1}{2} & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

\[
\frac{\partial y'}{\partial x'} = \frac{2}{3} \frac{\partial x'_0}{\partial x'} - \frac{1}{3} \frac{\partial x^2_0}{\partial x^2}
\]

\[
\frac{\partial y^2}{\partial x^2} = -\frac{2}{3} \frac{\partial x'_0}{\partial x'} + \frac{4}{3} \frac{\partial x^2_0}{\partial x^2}
\]

9. Sketch the curves y' = \frac{1}{2}, 1, 2 \& y^2 = 0, \pm \frac{1}{2}, \pm 1, \pm 2.

10. Sketch the tangent vectors \( \frac{\partial y'}{\partial p_0} \text{ and } \frac{\partial y^2}{\partial p_0} \text{ from } \mathbb{R} \).

11. Explain why \( \phi = (y', y^2) \) is not 1-1 on any open set around a point from 9.

Arbitrarily close to a point on L, you always have pairs of points with the same (y', y^2) - once reflection across L takes (x', x^2) to (x^2, x').

so does not change (y', y^2).
2/ Convince me by a sketch that the Klein bottle has a never-zero vector field.

\[ \nabla \]

3/ Find a never-zero vector field on any odd-dimensional sphere \( S^{2n-1} \).

(\text{Hint:} \ S^{2n-1} = \{ \vec{x} = (x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in \mathbb{R}^{2n} : \|\vec{x}\| = 1 \} \). Then \( (S^{2n-1})_{\vec{x}} = \{ \vec{y} \in \mathbb{R}^{2n} : \vec{x} \cdot \vec{y} = 0 \} \). For each \( \vec{x} \in S^{2n-1} \), you must find such a \( \vec{y} \neq 0 \).

(\text{Hint:} \ Try it for \( S^1 \subset \mathbb{R}^2 \) first.)

\[ S^1 \subset \mathbb{R}^2: \nabla \]

\[ x^2 \]

\[ y^2 \]

\[ \nabla \]

\[ x^1 \]

\[ -x^2 \]

\[ x^1 \]

\[ S^{2n-1} \subset \mathbb{R}^{2n}: \text{at each } \vec{x} = (x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \]

Take:

\[ \vec{y} \vec{x} = (-x^2, x^1, \ldots, -x_{2n-1}, x_{2n}) \]

\[ \vec{y} \vec{x} \cdot \vec{x} = 0, \text{ so } \vec{y} \vec{x} \in (S^{2n-1})_{\vec{x}}. \]

\[ \|\vec{y} \vec{x}\| = 1, \text{ so } \vec{y} \vec{x} \neq 0. \]