Math 481 H/wk 13, Due Wednesday, May 2

1. Let $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. We endow $M$ with the same orientation as $\mathbb{R}^2$ and with the Riemannian metric

$$g|_{(x,y)} = \frac{\langle , \rangle}{y^2}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^2$.

(a) Compute $R(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) \frac{\partial}{\partial x}$ and $g(R(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) \frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

(b) Prove that $M$ has constant sectional curvature $K = -1$.

2. Let $M$ be a smooth manifold with a smooth Riemannian metric $g$. Let $p \in M$. Let $X, Y$ be smooth vector fields such that $X(p), Y(p)$ are linearly independent nonzero vectors in $M_p$ spanning a 2-dimensional subspace $\sigma \leq M_p$.

Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha \delta - \beta \gamma \neq 0$. Put $X_1 = \alpha X + \beta Y$ and $Y_1 = \gamma X + \delta Y$.

Show that

$$g(R(X_1, Y_1)Y_1, X_1) = (\alpha \delta - \beta \gamma)^2 g(R(X, Y)Y, X)$$

and that

$$g(X_1, X_1)g(Y_1, Y_1) - g(X_1, Y_1)^2 = (\alpha \delta - \beta \gamma)^2 (g(X, X)g(Y, Y) - g(X, Y)^2).$$