8. Diffeomorphisms and Homeomorphisms

1. a) Manifolds $M$ and $N$ are **diffeomorphic** if there exists a one to one, onto map $F: M \rightarrow N$ such that both $F$ and $F^{-1}$ are smooth in patch coordinates.

b) That is, if $(U, \phi)$ is a patch at $p \in M$, and $(V, \psi)$ is a patch at $F(p) \in N$, then $(\psi \circ F \circ \phi^{-1})$ and $(\phi \circ F^{-1} \circ \psi^{-1})$ are smooth.

c) You only have to check this for one choice of an atlas on $M$ and an atlas on $N$. Then it holds for all other compatible charts.

d) **Note:** If we substitute “continuous” for “smooth” in our definitions of a manifold and diffeomorphism, we get the definitions of “topological manifold” and “homeomorphism”. Thus, diffeomorphic manifolds are homeomorphic.

e) An $n$-dimensional topological manifold, $n \leq 3$, can be given a unique differentiable structure. But if $n \geq 4$, there are topological manifolds that cannot be given a smooth structure.

2. **Example:** $\mathbb{R}^3, S^3, S^2 \times S^1, S^4 \times S^1, P^3, SO(3)$: all of dimension 3.

a) There are “topological invariants” that show every pair except the last are not homeomorphic.

An “extra problem” will be to show that $P^3$ and $SO(3)$ are homeomorphic.

b) At first you might think that $SO(3)$ and $S^2 \times S^1$ are homeomorphic: just choose $\vec{e}_1 \in S^2$. Then for each $\vec{e}_1$, choose $\vec{e}_2 \in S^1$. What is wrong with this argument?
Math 481
9. The Flow of a Vector Field

1. Flow Lines of a Vector Field \([\text{Frankel, pp. 30-35}]\)

   a) If \(Y\) is a vector field on \(M\), then an integral curve (or flow line) of \(Y\) is a curve \(\gamma(t)\) with \(\frac{d\gamma}{dt} = Y_{\gamma(t)}\).

   b) In a coordinate patch, if \(Y = \sum_{i=1}^{n} Y_i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i}\), then \(\gamma(t) = (x^1(t), \ldots, x^n(t))\) is obtained by solving the system of ODEs

   \[
   \frac{dx^i}{dt} = Y_i(x^1(t), \ldots, x^n(t)), \quad 1 \leq i \leq n.
   \]

   c) Example: \(M = \mathbb{R}^2, Y = \frac{\partial}{\partial x^1} + (x^2)^2 \frac{\partial}{\partial x^2}\).

   So, \(Y^1(x^1, x^2) = 1\) and \(Y^2(x^1, x^2) = (x^2)^2\). The flow lines \(\gamma(t) = (x^1(t), x^2(t))\) are solutions of \(\frac{dx^1}{dt} = 1, \frac{dx^2}{dt} = (x^2)^2\). Thus,

   \[
   x^1 = t + a, \quad x^2 = \frac{1}{t - b}
   \]

   is the flow line satisfying \(\gamma(0) = (a, b), b \neq 0\) and

   \[
   x^1 = t + a, x^2 = 0
   \]

   is the flow line satisfying \(\gamma(0) = (a, 0)\).

   d) Theorem: There is a unique maximal (i.e. defined on the largest possible \(t\) interval) integral curve \(\gamma_p\) with \(\gamma_p(0) = p\).

2. Local Flows

   a) Let \(\phi_t(p) = \gamma_p(t)\) (if the latter exists). This map moves every point of \(M\) along a flow line of \(Y\) for a time \(t\).

   b) It may not be possible to flow every point for a given time \(t \neq 0\) (see the above example). But, every point \(p\) lies in a chart neighborhood \(U\) that can be flowed for some (possibly small) fixed time; i.e. \(\phi_t(q)\) exists for all \(q \in U\) and \(-\epsilon < t < \epsilon\) (this is the Fundamental Theorem on p. 32 of Frankel: we say "local flows exist").

3. Local Flows of \(\gamma\) at a nonsingular point (i.e. \(Y_p \neq 0\))

   a) Theorem: If \(Y_p \neq 0\), there is a chart \((U, (y^1, \ldots, y^n))\) around \(p\) such that \(\frac{\partial}{\partial y^1}|_p = Y_q\) for all \(q \in U\).
1. a) A smooth map \( F : M^m \rightarrow N^n \) determines a linear \textit{tangent map} of tangent spaces
\[
F_* : T_p M \rightarrow T_{F(p)} N.
\]

b) For coordinates \( \phi = (x^1, \ldots, x^m) \) at \( p \in M \), and \( \psi = (y^1, \ldots, y^n) \) at \( F(p) \in N \),
\[
F_* \left( \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.
\]
i.e. the matrix of the linear transformation \( F_* \) is the Jacobian
\[
\frac{\partial (y^1, \ldots, y^n)}{\partial (x^1, \ldots, x^m)} = D(\psi \circ F \circ \phi^{-1}).
\]
c) We can check directly that the definition of \( F_* \) does not depend on the choice of coordinates.

Or, we can find a coordinate independent definition. Let \( \gamma(t) \) be a curve in \( M \) and \( \frac{d\gamma}{dt} \) a tangent vector tangent to \( \gamma(t) \). Define
\[
F_* \left( \frac{d\gamma}{dt} \right) = \frac{d}{dt} F(\gamma(t)).
\]

Then the formula above for \( F_* \left( \frac{\partial}{\partial x^j} \right) \) follows directly from the chain rule. Explain:
2. Example: $F : \text{upper half plane} \rightarrow \text{upper half plane}$, $F(x^1, x^2) = (y^1, y^2)$ where $y^1 = \frac{x^1}{x^2}$ and $y^2 = x^2$. We have

$$F_* \frac{\partial}{\partial x^1} = \frac{\partial y^1}{\partial x^1} \frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^1} \frac{\partial}{\partial y^2} = \frac{1}{x^2} \frac{\partial}{\partial y^1}$$  \hspace{1cm} (1)

$$F_* \frac{\partial}{\partial x^2} = \frac{\partial y^1}{\partial x^2} \frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^2} \frac{\partial}{\partial y^2} = -\frac{x^1}{(x^2)^2} \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}. \hspace{1cm} (2)$$

Or we may view equations (1) and (2) above as

$$(F_* \frac{\partial}{\partial x^1}, F_* \frac{\partial}{\partial x^2}) = \left( \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right) DF = \left( \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right) \left( \frac{\partial (y^1, y^2)}{\partial (x^1, x^2)} \right).$$

(This is the same type of formula as you used in Worksheet #4, just a slightly different viewpoint - maps instead of coordinate changes.)

So, if $p : (x^1, x^2) = (1, 1)$, then $F(p) = (y^1, y^2) = (10, 1)$ and

$$F_* \frac{\partial}{\partial x^1} \big|_p = 10 \frac{\partial}{\partial y^1} \big|_{F(p)}$$

$$F_* \frac{\partial}{\partial x^2} \big|_p = -100 \frac{\partial}{\partial y^1} \big|_{F(p)} + \frac{\partial}{\partial y^2} \big|_{F(p)}.$$

Locate the basis vectors $\frac{\partial}{\partial y^1} \big|_{F(p)}$ and $\frac{\partial}{\partial y^2} \big|_{F(p)}$: 

\[ \text{Diagram} \]