Math 481

3. Checking the manifold definition

1. Example: An atlas on \( P^1 \). Recall that \( P^1 \) is the space of all lines in \( \mathbb{R}^2 \) through \((0,0)\).

The map \( \phi_1 \) is defined on what set \( U_i \subset P^1 \)? Is \( (U_i, \phi_i) \) a coordinate patch? Are \( (U_1, \phi_1), (U_2, \phi_2) \) an atlas?

- Is every line through \((0,0)\) contained in some \( U_i \)?
- Is \( \phi_1(U_1 \cap U_2) \) open in \( \mathbb{R}^2 \)? \( \phi_2(U_1 \cap U_2) \)?
- What is \( \phi_2 \circ \phi_1^{-1} \)? \( \phi_1 \circ \phi_2^{-1} \)?

2. Similarly, on Worksheet 1, for \( P^2 \) we have

- \( \phi_1(L) = (u_1^1, u_2^1) \),
- \( \phi_2(L) = (u_1^2, u_2^2) \),
- \( \phi_3(L) =? \)

3. Example: Consider \( k \) numbered rods of length 1, in a closed chain with hinged joints, in the plane. What is the configuration space if \( k = 3 \),

a) assuming the first rod is fixed,

b) no constraints assumed?
Math 481
4. Orientability

1. A manifold is orientable if it has an atlas such that whenever \( U_i \cap U_j \neq \emptyset \), then \( D(\phi_i \circ \phi_j^{-1}) \) has positive determinant.

   a) Here, \( \phi_i = (u_i^1, \ldots, u_i^n): U_i \to \mathbb{R}^n \) and \( D(\phi_i \circ \phi_j^{-1}) \) is the Jacobian

   \[
   \begin{pmatrix}
   \frac{\partial u_i^1}{\partial u_j^1} & \cdots & \frac{\partial u_i^1}{\partial u_j^n} \\
   \vdots & \ddots & \vdots \\
   \frac{\partial u_i^n}{\partial u_j^1} & \cdots & \frac{\partial u_i^n}{\partial u_j^n}
   \end{pmatrix}
   \]

   which we also write as

   \[
   \frac{\partial (u_i^1, \ldots, u_i^n)}{\partial (u_j^1, \ldots, u_j^n)}
   \]

   b) on \( U_i \cap U_j \), \( \phi_i \circ \phi_j^{-1} \) and \( \phi_j \circ \phi_i^{-1} \) are inverse maps, so, by the Multivariable Chain Rule, their Jacobians are inverse matrices. Hence, their determinants are inverse numbers (remember for \( n \times n \) matrices \( A, B \), \( \det(AB) = \det(A) \det(B) \)). Thus, these determinants are nonzero.

   c) Since \( \det D(\phi_i \circ \phi_j^{-1}) \neq 0 \), it has the same sign at all the points in a connected component of \( U_i \cap U_j \).

   d) Note: switching any pair of \( \phi_i \) coordinates switches two rows of \( D(\phi_i \circ \phi_j^{-1}) \). Switching any pair of \( \phi_j \) coordinates switches two columns of \( D(\phi_i \circ \phi_j^{-1}) \). Either switch changes the sign of the determinant.

2. A 2-dimensional manifold is orientable if and only if it does not contain a Möbius band. For example, \( \mathbb{P}^2 \) is not orientable.
3. a) Suppose $M$ is an orientable manifold and $\mathcal{A} = \{ (U_\alpha, \phi_\alpha = (u^1_\alpha, \ldots, u^n_\alpha)) \}$ is an orienting atlas ($\det \frac{\partial (u^1_\alpha, \ldots, u^n_\alpha)}{\partial (u^1_\beta, \ldots, u^n_\beta)} > 0$ on $U_\alpha \cap U_\beta$). For any other compatible coordinate patch $(U, \phi = (x^1, \ldots, x^n))$ with $U$ connected, the sign of $\det \frac{\partial (x^1, \ldots, x^n)}{\partial (u^1_\alpha, \ldots, u^n_\alpha)}$ is the same for all $U_\alpha$ with $U \cap U_\alpha \neq \emptyset$.

Reason: by the Multivariable Chain Rule, we have

$$\frac{\partial (x^1, \ldots, x^n)}{\partial (u^1_\beta, \ldots, u^n_\beta)} = \frac{\partial (x^1, \ldots, x^n)}{\partial (u^1_\alpha, \ldots, u^n_\alpha)} \frac{\partial (u^1_\alpha, \ldots, u^n_\alpha)}{\partial (u^1_\beta, \ldots, u^n_\beta)}$$

and $\det$ is multiplicative.

b) Conclusion: If a manifold $M$ has a finite atlas of connected coordinate patches, there is a finite recursive procedure for deciding if $M$ is orientable: Start with the first patch and try to alter all those that intersect it to get the determinant of the Jacobians to be positive. If this cannot be done, $M$ is not orientable. If it can, repeat starting with the second coordinate patch.

Reference: Frankel, pp. 82-85.