Math 481
14. Differential Forms

Reference: Frankel, pp. 66-69

1. a) A differential r-form \( \alpha \) on \( M \) is an antisymmetric r-covariant tensor field. At each \( p \in M \):

\[
\alpha(\ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots) = -\alpha(\ldots, \vec{v}_j, \ldots, \vec{v}_i, \ldots)
\]

b) Examples

(i) If \((U, (x^1, \ldots, x^n))\) is a patch, \(dx^1, \ldots, dx^n\) are a basis for the 1-forms at any point in \(U\).

(ii) If \( M = \mathbb{R}^3 \), the scalar triple product \( \alpha(\vec{v}, \vec{w}, \vec{u}) = \vec{v} \times \vec{w} \cdot \vec{u} \) is a 3-form on \( M \).

c) An \( r \)-form is determined at any point by its \( \binom{n}{r} \) components

\[
a_{i_1 < \ldots < i_r} = \alpha \left( \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}} \right).
\]

d) We write

\[
I = (i_1, \ldots, i_r), \quad |I| = r, \quad \frac{\partial}{\partial x^{i_j}} = \left( \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}} \right), \quad \rightarrow = (i_1 < \ldots < i_r).
\]

For \( J = (j_1, \ldots, j_r) \), set

\[
\delta^I_J = \begin{cases} 
1, & \text{if } J \text{ is an even permutation of } I \\
-1, & \text{if } J \text{ is an odd permutation of } I \\
0, & \text{otherwise}.
\end{cases}
\]

2. a) The wedge product \( \alpha \wedge \beta \) of an \( r \)-form \( \alpha \) and an \( s \)-form \( \beta \) is defined by

\[
(\alpha \wedge \beta)(\vec{v}_1, \ldots, \vec{v}_{r+s}) = \sum_{(K,L)} \delta^{(K,L)}_{(i_1, \ldots, i_r)} \alpha(\vec{v}_K) \beta(\vec{v}_L),
\]

where \( \Sigma \) is over all permutations \((K, L)\) of \((1, \ldots, r+s)\).

b) Example:

\[
(dx^1 \wedge dx^2) \left( \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) = \begin{cases} 
1 & \text{if } k = 1, l = 2 \\
-1 & \text{if } k = 2, l = 1 \\
0 & \text{otherwise}
\end{cases}
\]

c) Similarly, \( dx^I \left( \frac{\partial}{\partial x^j} \right) = \delta^I_J \).

d) In particular, \( dx^1 \wedge dx^2 = -dx^2 \wedge dx^1 \) and \( dx^i \wedge dx^j = -dx^j \wedge dx^i \).
3. a) Key fact:

Let $\alpha$ be an $r$-form on $M$. In a coordinate patch, write $\alpha(\partial/\partial x^I) = a_I$ for all $I = i_1 < \ldots < i_r$. Then

$$\alpha = a_{i_1 < \ldots < i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r} := a_I dx^I.$$

To prove this, note that the left hand side and the right hand side of the above equation do the same thing to any $\frac{\partial}{\partial x^I}$ since $dx^I \left( \frac{\partial}{\partial x^J} \right) = \delta^I_J$. By linearity, then, both sides are the same tensor.

b) From a) we conclude that the $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_r}, i_1 < \ldots < i_r$, form a basis for the $r$-forms at a point.

c) Therefore, the $r$-forms at a point form a vector space of dimension $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

4. Properties of $\wedge$:

a) Bilinear: $(a_1 \alpha_1 + a_2 \alpha_2) \wedge \beta = a_1 (\alpha_1 \wedge \beta) + a_2 (\alpha_2 \wedge \beta)$ etc.

b) if $\alpha$ is an $r$-form and $\beta$ is an $s$-form, then $\beta \wedge \alpha = (-1)^r s \alpha \wedge \beta$.

c) Associative: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = \alpha \wedge \beta \wedge \gamma$. 
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15. Pullback of Differential Forms

Pullback of an \( r \)-form (in fact, of any \( r \)-covariant tensor)

Reference: Frankel, pp. 77-82

For a map \( F : M^m \rightarrow N^n \), an \( r \)-covariant tensor \( \alpha \) on \( N \) pulls back to an \( r \)-covariant tensor \( F^* \alpha \) on \( M \):

a) Invariant definition of \( F^* \alpha \) at any \( p \in M \):

\[
(F^* \alpha)(\vec{u}_1, \ldots, \vec{u}_r) = \alpha(F_*\vec{v}_1, \ldots, F_*\vec{v}_r)
\]

i.e., \( \alpha \) pulls back because tangent vectors push forward!

b) But, vector fields on \( M \) do not push forward to vector fields on \( N \). For example, take \( M = \mathbb{R}^2, N = \mathbb{R}^2 \) and \( \vec{V} = \frac{\partial}{\partial t} \) a vector field on \( M \). Let \( F \) be the map pictured below:

Then, most points of \( N \) have no vector \( F_*\frac{\partial}{\partial t} \), and one point has two such vectors! The fact that \( r \)-forms pull back to \( r \)-forms is a major advantage!

![Diagram](image)

In coordinates, \((u^1, \ldots, u^m)\) on \( M \) and \((x^1, \ldots, x^n)\) on \( N \):

\[
F(u^1, \ldots, u^m) = (x^1(u^1, \ldots, u^m), \ldots, x^n(u^1, \ldots, u^m))
\]

\[
F^* dx^j = \frac{\partial x^j}{\partial u^i} du^i
\]

\[
F^* \left( \sum_{i,j} a_{ij} dx^i \right) = \sum_{i,j \leq 1 \leq i, \ldots, r \leq m} a_{ij} \left( \frac{\partial x^j_1}{\partial u^i_1} du^i_1 \right) \wedge \ldots \left( \frac{\partial x^j_r}{\partial u^i_r} du^i_r \right)
\]

where \( J = (1 \leq j_1, \ldots, j_r \leq n) \).

d) Example: Let \( \alpha = V_i(x^1, x^2, x^3) dx^i \) be a 1-form on \( \mathbb{R}^3 \) (here \( x^1, x^2, x^3 \) are the standard coordinates on \( \mathbb{R}^3 \)). Let \( F : \mathbb{R} \rightarrow \mathbb{R}^3 \) be a curve: \( F(t) = (x^1(t), x^2(t), x^3(t)) \). Then,

\[
F^* \alpha = V_i(x^1(t), x^2(t), x^3(t)) \frac{dx^i}{dt} dt,
\]

a 1-form on \( \mathbb{R} \).

If \( \vec{V} = V_i(x^1, x^2, x^3) \frac{\partial}{\partial x^i} \) is regarded as a force vector field in \( \mathbb{R}^3 \), the work done by \( \vec{V} \) on a particle moving along the curve \( F \), \( t_0 \leq t \leq t_1 \) is

\[
\int_{t_0}^{t_1} \vec{V}_{F(t)} \cdot \frac{dF}{dt} dt = \int_{t_0}^{t_1} V_i(x^1(t), x^2(t), x^3(t)) \frac{dx^i}{dt} dt
\]

\[
= \int_{t_0}^{t_1} F^* \alpha
\]
• a) Pullback is not a differential operator. It is a linear operator at each point:

\[ F^*(a\alpha + b\beta) = aF^*\alpha + bF^*\beta. \]

and it also satisfies

\[ F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta) \]

b) Key Fact: For any \( r \)-form \( \alpha \), \( F^*(d\alpha) = d(F^*\alpha) \).
16. Singular $r$-chains

1. a) A "singular $r$-cube" $\gamma$ in $M$ is a smooth map $\gamma : [0, 1]^r \to M$.
   
   b) A "singular $r$-chain" is a formal linear combination of singular $r$-cubes: $\sum_{i=1}^{d} a_i \gamma_i$ where $a_i \in \mathbb{R}$.

2. Given a singular 1-cube $\gamma$ in $\mathbb{R}^3$, we want to define the boundary of $\gamma$ (denoted $\partial \gamma$) to be $\gamma(1) - \gamma(0)$ (a singular 0-chain).

3. Given a singular 2-cube $\gamma$ in $\mathbb{R}^3$, by looking at what $\gamma$ does to the edges of the square, we get four 1-cubes, each oriented by increasing $u^i$. We want $\partial \gamma$ to be
   
   $\gamma_{2,0} + \gamma_{1,1} - \gamma_{2,1} - \gamma_{1,0}$.

4. For any $r$-cube $\gamma : [0, 1]^r \to M$, define:
   
   a) $(r-1)$-cubes $\gamma_{i, \alpha} : [0, 1]^{r-1} \to M$, $1 \leq i \leq r$, $\alpha = 0$ or 1, by
   
   $\gamma_{i, \alpha}(u^1, \ldots, u^{r-1}) = \gamma(u^1, \ldots, u^{i-1}, \alpha, u^i, \ldots, u^{r-1})$,

   b) the boundary of $\gamma$ by
   
   $\partial \gamma = \sum_{i, \alpha} (-1)^{i+\alpha} \gamma_{i, \alpha}$.

   (If $r = 0$, set $\partial \gamma = 1 \in \mathbb{R}$.)

5. **Theorem**: For any $r$-cube $\gamma$, $\partial^2 \gamma = 0$.