11. Dual Space and Differential of a Function

1. a) Define $M^*_p = \text{dual space of } M_p = \text{all linear functionals } \alpha : M_p \rightarrow \mathbb{R}.$
   We call $\alpha$ a covector or a 1-form.

   b) In a coordinate patch:
   - Basis of $M_p : \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$.
   - Basis of $M^*_p : \{dx^1, \ldots, dx^n\}$ where "$dx^i$" is defined by
     \[
     dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta^i_j, \text{ and so}
     \]
     \[
     dx^i \left( \sum_j v^j \frac{\partial}{\partial x^j} \right) = v^i \text{ by linearity.}
     \]

2. a) The differential of a function $f : M \rightarrow \mathbb{R}$ is a 1-form $df : M_p \rightarrow \mathbb{R}$ defined by
   \[ df(v) = v(f). \]

   b) In a coordinate patch, this just says
   \[
   df \left( \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} \right) = \sum_{j=1}^{n} v^j df \left( \frac{\partial}{\partial x^j} \right)
   = \sum_{j=1}^{n} v^j \frac{\partial f}{\partial x^j}
   \]

   c) Conclusion: $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$ The proof of this is to apply both sides to a basis vector $\frac{\partial}{\partial x^i}.$ Since they give the same answer, namely $\frac{\partial f}{\partial x^i},$ they are the same linear functionals.

   d) Note: If we identify $\mathbb{R}$ with its tangent space (by identifying $y \in \mathbb{R}$ with $y \frac{\partial}{\partial y} \in \mathbb{R}_y$), then for $f : M \rightarrow \mathbb{R},$ the tangent map $f_* : M_p \rightarrow \mathbb{R}_{f(y)}$ is the same as $df : M_p \rightarrow \mathbb{R}.$ To see this: For a chart on $M,$ write $y = f(x^1, \ldots, x^n).$ Then,
   \[
   df \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial y}{\partial x^i}
   \]
   and
   \[
   f_* \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial y}{\partial x^i} \frac{\partial}{\partial y}.
   \]
12. Tensors

1. a) A tensor \( W \), \( r \) times covariant and \( s \) times contravariant, at \( p \) is a multilinear functional

\[
W : M_p^* \times \ldots \times M_p^* \times M_p \times \ldots \times M_p \to \mathbb{R}
\]

where there are \( s M_p^* \) terms and \( r M_p \) terms in the domain.

b) A once-covariant tensor is a 1-form. A once-contravariant tensor is a linear map \( M_p^* \to \mathbb{R} \), so lies in \( (M_p^*)^s = M_p \) and hence is a tangent vector at \( p \).

c) The components of \( W \) with respect to a patch \( (U, \phi = (x^1, \ldots, x^n)) \) are

\[
W_{j_1 \ldots j_r}^{i_1 \ldots i_s} = W \left( dx^{i_1}, \ldots, dx^{i_s}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_r}} \right)
\]

d) If \( W \) is defined for all \( p \in M \), then for each chart \( (U, \phi) \) we get \( n^{r+s} \) component functions on \( U \). If these are smooth, we say \( W \) is a tensor field on \( M \).

e) By multilinearity,

\[
W \left( \sum_{i_1=1}^{n} a_{i_1} dx^{i_1}, \ldots, \sum_{i_s=1}^{n} a_{i_s} dx^{i_s}, \sum_{j_1=1}^{n} v_{j_1} \frac{\partial}{\partial x^{j_1}}, \ldots, \sum_{j_r=1}^{n} v_{j_r} \frac{\partial}{\partial x^{j_r}} \right)
\]

\[
= \sum_{1 \leq i_1, \ldots, i_s, j_1, \ldots, j_r \leq n} a_{i_1} \ldots a_{i_s} v_{j_1} \ldots v_{j_r} W_{j_1 \ldots j_r}^{i_1 \ldots i_s}
\]

where we add (from 1 to \( n \)) on every index that appears once up and once down.

f) The Einstein summation convention omits the "\( \sum \)"s.

2. If we change patches, say to \( (U', (x'^1, \ldots, x'^n)) \), then on \( U \cap U' \) the components of \( W \) with respect to the charts satisfy this transformation law (by e) on the previous lecture):

\[
W_{k \ldots j}^{i_1 \ldots i_s} = W \left( dx^{i_1}, \ldots, dx^{i_s}, \frac{\partial}{\partial x'^k}, \ldots, \frac{\partial}{\partial x'^l} \right)
\]

\[
= W \left( \frac{\partial x'^i}{\partial x^a} dx^a, \ldots, \frac{\partial x'^j}{\partial x^a} dx^a, \frac{\partial x^c}{\partial x'^k}, \ldots, \frac{\partial x^d}{\partial x'^l} \right)
\]

\[
= \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^j}{\partial x^b} \frac{\partial x^c}{\partial x'^k} \ldots \frac{\partial x^d}{\partial x'^l} W_{c \ldots d}^{a \ldots b}
\]