For which points \( p \) on this torus do \((x, y)\) NOT give coordinates around \( p \)? Mark the points \( p \).

Projection to \( xy \)-plane \( \text{NOT} \ 1-1 \)

around points \( p \) on the outer & inner circles

A 2-dimensional manifold \( M \) has a coordinate patch \( U_1(u^1, u^2) \) \( \{ u^1 > 0 \} \)

and a map \( F: U \rightarrow \mathbb{R}^3 \) satisfying

\[
\begin{align*}
x^1 &= \cos(u^2), & x^2 &= \sin(u^2), & x^3 &= (u^1)^2.
\end{align*}
\]

\( a \) Express the \( F \) in terms of the \( \frac{\partial}{\partial x^i} \) at the point \( p: u^1 = 1, u^2 = 0 \).

\[
\begin{align*}
F_x \frac{\partial}{\partial u^1} &= \frac{\partial x^1}{\partial u^1} \frac{\partial}{\partial x^1} = 2u^1 \frac{\partial}{\partial x^1}. & \text{At } u^1 = 1, u^2 = 0: 2 \frac{\partial}{\partial x^1}
\end{align*}
\]

\[
\begin{align*}
F_x \frac{\partial}{\partial u^2} &= \frac{\partial x^1}{\partial u^2} \frac{\partial}{\partial x^1} = -\sin u^2 \frac{\partial}{\partial x^1} + \cos u^2 \frac{\partial}{\partial x^2}. & \text{At } u^1 = 1, u^2 = 0: \frac{\partial}{\partial x^2}
\end{align*}
\]

\( b \) Sketch \( M \) in \( \mathbb{R}^3 \), marking the coordinate curves \( u^1 = 1 \) & \( u^2 = 0 \) through \( p: (u^1, u^2) = (1, 0) \), and the tangent vectors \( F_x \frac{\partial}{\partial u^i} \) at \( p \).

\( p = (\cos \theta, \sin \theta, 1) = (1, 0, 1) \)
A rod moves freely in the plane. What is the configuration space? What is the dimension of this space?

$$IR^2 \times S^1$$

dimension 3

$$M = \{(x,y,z,u,v) \in IR^5 : \frac{x^2+y^2+z^2+u^2+v^2}{4} = \frac{4x^2+y^2+4z^2+4u^2+v^2}{4} = 1 \}$$

(a) Use the IFT to show $M$ is a manifold.

$$F : IR^5 \rightarrow IR^2 : F(x,y,z,u,v) = (x^2+y^2+z^2+u^2+v^2 - 1, 4x^2+y^2+4z^2+4u^2+v^2 - 1)$$

$$DF = \begin{pmatrix} 12x & 24y & 2z & 2u & 2v \\ 8x & 8y & 8z & 8u & 8v \end{pmatrix}$$

The 2 equations defining $M$ show that $v \neq 0$ at points in $M$.

Similarly, not all of $x, y, z, u$ can be $= 0$.

$\exists x \neq 0, \det \begin{pmatrix} 2x & 2u \\ 8x & v/2 \end{pmatrix} = xv - 16xu \neq 0$; similarly $y \neq 0$ if $v \neq 0$. If $u \neq 0$,

rank $DF = 2$.

Note: You must use the definition of $M$ to show the rows are linearly independent — for other values of $(x,y,z,u,v)$, they can be dependent!

(b) Give a choice from $(x,y,z,u,v)$ that can be used as coordinates around $p = (\frac{2}{\sqrt{3}}, 0, 0, 0, \frac{2}{\sqrt{3}}) \in M$. Explain.

$$(DF)(p) = \begin{pmatrix} 4x & 0 & 0 & 0 & 4/\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 4/\sqrt{3} & 4/\sqrt{3} \\ 0 & 0 \end{pmatrix} = 0$$

$(y,z,u)$ can be used as coordinates on $M$ around $p$.  

a) Convince me by a picture that on a 2-torus $T^2 = S^1 \times S^1$ there does exist a nonvanishing continuous vector field

\[ T^2 = \begin{array}{c}
\text{square} \\
\text{torus}
\end{array} = \begin{array}{c}
\text{nonvanishing vector field}
\end{array} \]

b) Use a) to explain why $T^2$ and $S^2$ are not diffeomorphic. The tangent map of a diffeomorphism takes a continuous, nonvanishing vector field to a continuous, nonvanishing vector field. Since $T^2$ has such a field but $S^2$ does not, they are not diffeomorphic.

c) Let $M$ be a 2-manifold. Let $(U, \psi = (x,y))$ and $(U', \psi' = (z,w))$ are coordinate patches on $M$. Write \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial w} \) in terms of \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \):

\[ \frac{\partial}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial w} = \frac{\partial x}{\partial w} \frac{\partial}{\partial x} + \frac{\partial y}{\partial w} \frac{\partial}{\partial y} \]
Let \( \mathbb{P}^2 \) be the projective plane, that is, the set of all lines through the origin in \( \mathbb{R}^3 \).

Consider the function
\[
\hat{T} : \mathbb{P}^2 \to \mathbb{R}
\]
\[
\hat{T}([x:y:z]) = \frac{2x}{x+y+z}
\]

(a) Consider the chart \((U, \psi)\)
where \( U = \{ [x:y:z] \in \mathbb{P}^2 \mid z \neq 0 \} \)
\[
\psi([x:y:z]) = \left( \frac{x}{z}, \frac{y}{z} \right)
\]

Compute \( d\hat{T} \) explicitly in this chart
\[
\psi^{-1}(u', u^2) = [u':u^2:1]
\]
\[
(\hat{T} \circ \psi^{-1})(u', u^2) = \frac{2u'}{u'+u^2+1}
\]
\[
d\hat{T} = \frac{\partial \hat{T}}{\partial u'} du' + \frac{\partial \hat{T}}{\partial u^2} du^2 =
\]
\[
= \frac{2u'^2 + u'^2}{(u'+u^2+1)^2} du' - \frac{2u'}{(u'+u^2+1)^2} du^2
\]

(b) Consider the map \( \delta : \mathbb{R}^+ \to \mathbb{P}^2 \)
given by \( \delta(t) = [1:t^2:t] \), \( t > 0 \)
compute \( \delta_* \left( \frac{\partial}{\partial t} \right) \) explicitly in the above chart.
\((\varphi \circ \sigma)(t) = \left( \frac{1}{t}, \frac{t^2}{t} \right) = \left( \frac{1}{t}, t \right)\)

\(\xi = \left( \frac{\partial}{\partial t} \right) = \frac{\partial u^1}{\partial t} \frac{\partial}{\partial u^1} + \frac{\partial u^2}{\partial t} \frac{\partial}{\partial u^2} = \)

\[-\frac{1}{t^2} \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2}\]