Math 427 Midterm Exam 3 (Solutions)
Due in class on Wednesday, November 19, 2014

PRINT YOUR NAME:

Problem 1. [10 points]
Consider the ideal \( I = (2x^2, x^3) \triangleleft \mathbb{Z}[x] \). Prove that \( I \) is not a principal ideal.

Solution.
Suppose, on the contrary, that \( I \) is a principal ideal, so that there exists \( f \in \mathbb{Z}[x] \) such that \( I = (f) \triangleleft \mathbb{Z}[x] \). Since \((f) = (-f)\), after possibly replacing \( f \) by \(-f\) we may assume that the leading coefficient of \( f \) is \( > 0 \).

Since both \( 2x^2 \) and \( x^3 \) are divisible by \( x^2 \), it follows that every element of \( I = (2x^2, x^3) \) is divisible by \( x^2 \) as well. Since \( f \in I \), it follows that \( f = x^2f_0 \) for some \( f_0 \in \mathbb{Z}[x] \). Note that \( f_0 \neq 0 \) since \( I = (f) \neq \{0\} \). In particular, it follows that every nonzero element of \( I \) has degree \( \geq \deg(f) \geq 2 \). Since \( 2x^2 \in I \) and \( \deg(2x^2) = 2 \), this implies that \( \deg(f) = 2 \). Therefore \( \deg(f_0) = 0 \), so that \( f_0 = a \in \mathbb{Z} \) for some integer \( a > 0 \).

Then \( f = ax^2 \) and \( I = (ax^2) \). Since \( x^3 \in I \), there exists a nonzero \( h \in \mathbb{Z}[x] \) such that \( x^3 = fh = ax^2h \). Note that \( 3 = \deg(x^3) = \deg(ax^2) + \deg(h) = 2 + \deg(h) \), so that \( \deg(h) = 1 \). Thus \( h = bx + c \) for some \( b, c \in \mathbb{Z} \) with \( b \neq 0 \).

Hence from \( x^3 = ax^2(bx + c) = abx^3 + ac \) we get \( ab = 1 \) and \( ac = 0 \). Since \( a \neq 0 \), it follows that \( c = 0 \). Thus \( h = bx \) where \( ab = 1 \). Since \( a, b \in \mathbb{Z} \) and \( a > 0 \), it follows that \( a = b = 1 \). Thus \( f = ax^2 = x^2 \), \( I = (f) = (x^2) \) and \( x^2 \in I \).

However, both \( 2x^2 \) and \( x^3 \) have all the coefficients at \( x^n \) with \( n = 0, 1, 2 \) being even integers. Therefore for every element \( g \in I = (2x^2, x^3) \), \( g = b_0 + b_1x + b_2x^2 + \ldots \), the coefficient \( b_2 \) is even. Hence \( x^2 \notin I \), yielding a contradiction.

Problem 2. [10 points]
Consider the product ring \( R = \mathbb{Z} \times \mathbb{Z} \).

(a) [6 points] Give an example of a maximal ideal \( I \triangleleft R, I \neq R \).
(b) [6 points] Give an example of a non-maximal ideal \( I \triangleleft R, \{0\} \neq I \neq R \).

Justify that your examples have the required properties.

Solution.

(a) The set \( I = 2\mathbb{Z} \times \mathbb{Z} \) is easily seen to be an ideal in \( R = \mathbb{Z} \times \mathbb{Z} \). We claim that \( I \) is a maximal ideal in \( R \).

Consider the map \( \phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_2 \) given by \( \phi(m, n) = [m]_2 \), for \( m, n \in \mathbb{Z} \).

Then \( \phi \) is a surjective ring homomorphism with \( \ker(\phi) = I \). Thus the rings \( R/I \) and \( \mathbb{Z}_2 \) are isomorphic, by the First Isomorphism Theorem. The ring \( \mathbb{Z}_2 \) is a field and hence \( R/I \) is also a field. The fact that \( R/I \) is a field implies that \( I \) is a maximal ideal in \( R \).
(b) Consider the set \( I' = 2\mathbb{Z} \times \{0\} \subseteq \mathbb{Z} \times \mathbb{Z} \). It is easy to see that \( I' \) is an ideal in \( R \). Moreover, for the ideal \( I = 2\mathbb{Z} \times \mathbb{Z} \) as in part (a), we have

\[ I' \subseteq I \subseteq R \]

and \( I' \neq I, I \neq R \). Therefore the ideal \( I' \) is not maximal in \( R \).

**Problem 3.** [10 points]

Prove that the rings \( \mathbb{Q}[x]/(x - 5) \) and \( \mathbb{Q} \) are isomorphic.

**Hint:** Use the First Isomorphism Theorem for rings.

**Solution.**

Indeed consider the map \( \phi : \mathbb{Q}[x] \to \mathbb{Q} \) defined as \( \phi(f) := f(5) \) for every \( f(x) \in \mathbb{Q}[x] \). Then \( \phi \) is a ring homomorphism.

Moreover, \( \phi \) is surjective, since for every \( r \in \mathbb{Q} \), when we view \( r \) as a polynomial of degree 0 in \( \mathbb{Q}[x] \), we have \( f(r) = r \).

We claim that \( \ker(\phi) = (x - 5) \triangleleft \mathbb{Q}[x] \).

Indeed, if \( f \in (x - 5) \) then \( f = a(x)(x - 5) \) for some \( a(x) \in \mathbb{Q}[x] \) and \( \phi(f) = a(5)(5 - 5) = 0 \), so that \( f \in \ker(\phi) \). Thus \( (x - 5) \subseteq \ker(\phi) \).

Suppose now that \( f \in \ker(\phi) \) is arbitrary. By performing division with remainder in \( \mathbb{Q}[x] \) we have \( f = q(x - 5) + r \) for some \( q, r \in \mathbb{Q}[x] \) such that \( \deg(r) < \deg(x - 5) = 1 \). Hence \( \deg(r) = 0 \), so that \( r \in \mathbb{Q} \) is a constant.

We have \( r = f - q(x - 5) \). Since \( f \in \ker(\phi) \), it follows that \( \phi(f) = f(5) = 0 \) and therefore

\[ r = \phi(r) = f(5) - q(5)(5 - 5) = 0. \]

Thus \( r = 0 \) and \( f = q(x - 5) \), so that \( f \in (x - 5) \). Since \( f \in \ker(\phi) \) was arbitrary, it follows that \( \ker(\phi) \subseteq (x - 5) \).

Since we already know that \( (x - 5) \subseteq \ker(\phi) \), it follows that \( \ker(\phi) = (x - 5) \triangleleft \mathbb{Q}[x] \), as claimed.

Thus \( \phi : \mathbb{Q}[x] \to \mathbb{Q} \) is a surjective ring homomorphism with \( \ker(\phi) = (x - 5) \).

Therefore the rings \( \mathbb{Q}[x]/(x - 5) \) and \( \mathbb{Q} \) are isomorphic, by the First Isomorphism Theorem.

**Problem 4.** [10 points]

Prove that if \( G \) is a group of order 56, then \( G \) has a normal Sylow \( p \)-subgroup for some prime \( p \) dividing the order of \( G \).

**Solution.**

We have \( |G| = 56 = 7 \cdot 8 = 7 \cdot 2^3 \).

Let \( n_7 \) be the number of Sylow 7-subgroups in \( G \). Then, by the 3-d Sylow Theorem, \( n_7 | 8 \) and \( n_7 \equiv 1 \) mod 7. It follows that either \( n_7 = 1 \) or \( n_7 = 8 \).

If \( n_7 = 1 \), then \( G \) has a unique Sylow 7-subgroup, which is therefore normal, as required.

Suppose now that \( n_7 = 8 \). Every Sylow 7-subgroup of \( G \) has order 7, and any two such subgroups are either equal or their intersection is \( \{1\} \).

Therefore the number of elements of order 7 in \( G \) is equal to \((7 - 1)n_7 = 6 \cdot 8 = 48\).
Hence $G$ has $56 - 48 - 1 = 7$ nontrivial elements $a_1, \ldots, a_7$ of orders different from 7.

By the First Sylow Theorem, $G$ does possess at least one Sylow 2-subgroup. Every Sylow 2-subgroup $P$ of $G$ has order 8 and thus $P$ consists of the identity element 1 and of 7 nontrivial elements whose orders are divisors of 8, i.e. whose orders are nonzero powers of 2. Since the only nontrivial elements of $G$ with orders different from 7 are the elements $a_1, \ldots, a_7$, it follows that $P \subseteq \{1, a_1, \ldots, a_7\}$. Since $|P| = 8$, it follows that $P = \{1, a_1, \ldots, a_7\}$. Thus $G$ has a unique Sylow 2-subgroup $P$, and therefore $P$ is normal on $G$.

**Problem 5.** [10 points]

Prove that if $R$ is an integral domain then the ring of formal powers series $R[[x]]$ is also an integral domain.

**Solution.**

To show that $R[[x]]$ is an integral domain, we need to verify that if $f, g \in R[[x]]$ are such that $f \neq 0$ and $g \neq 0$ in $R[[x]]$ then $fg \neq 0$ in $R[[x]]$.

Thus let $f, g \in R[[x]]$ be such that $f \neq 0$ and $g \neq 0$ in $R[[x]]$.

Then $f = a_mx^m + a_{m+1}x^{m+1} + \ldots$ and $g = b_nx^n + b_{n+1}x^{n+1} + \ldots$ for some $n, m \geq 0$, $a_i, b_j \in R$ such that $a_m \neq 0$ and $b_n \neq 0$ in $R$. Therefore

$$fg = a_mb_nx^{n+m} + (a_mb_{n+1} + a_{m+1}b_n)x^{m+n+1} + \ldots$$

Since $a_m \neq 0$, $b_n \neq 0$ and since, by assumption, $R$ is an integral domain, it follows that $a_mb_n \neq 0$ in $R$. Hence $fg \neq 0$ in $R[[x]]$, as required.