Problem 1. [10 points]
Consider the subgroup
\[ H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, ac \neq 0 \right\} \]
of $GL(2, \mathbb{R})$ (you do not have to prove that $H$ is a subgroup of $GL(2, \mathbb{R})$ and can take this fact for granted).

(1) [8 points] Find an infinite sequence of distinct elements $g_n \in GL(2, \mathbb{R})$, where $n = 1, 2, \ldots$, such that whenever $i \neq j, i, j \geq 1, i, j \in \mathbb{Z}$, then $g_iH \neq g_jH$.

Prove that your sequence has the required property.

(2) [2 points] Explain what the result of part (1) implies regarding the index $[GL(2, \mathbb{R}) : H]$.

Solution.

(1) Let $g = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. For $n \geq 1$ put $g_n = g^n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$. We claim that this sequence has the required properties. Indeed, for any integers $i, j \geq 1$ such that $i \neq j$ we have $g_i^{-1}g_j = g^{i-j} = \begin{bmatrix} 1 & 0 \\ j-i & 1 \end{bmatrix} \notin H$ and therefore $g_iH \neq g_jH$.

(2) Since in (1) we have found infinitely many distinct cosets $g_nH, n = 1, 2, \ldots$ of $H$ in $GL(2, \mathbb{R})$, it follows that $[GL(2, \mathbb{R}) : H] = \infty$.

Problem 2. [10 points]
Consider
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 7 & 1 & 3 & 5 \end{pmatrix} \in S_7. \]

(1) [5 points] Find $\epsilon(\sigma)$.

(2) [5 points] Express $\sigma$ as a product of transpositions.

Solution.

(1) First we find the cycle decomposition for $\sigma$, namely $\sigma = (1 \ 2 \ 4 \ 7 \ 5)(3 \ 6)$. Hence, by multiplicativity of $\epsilon$, we have
\[ \epsilon(\sigma) = (-1)^{5-1}(-1)^{2-1} = -1. \]

(2) Using the fact, stated in class, that for $k \geq 2$ we have $(a_1 \ a_2 \ldots a_k) = (a_1 \ a_k)(a_1 \ a_{k-1})\ldots(a_1 \ a_2)$, we get:
\[ \sigma = (1 \ 2 \ 4 \ 7 \ 5)(3 \ 6) = (1 \ 5)(1 \ 7)(1 \ 4)(1 \ 2)(3 \ 6). \]

Problem 3. [10 points]

(1) [5 points] For the subgroup $H = \langle s \rangle \leq D_{24}$ find the index $[D_{24} : H]$.

(2) [5 points] Prove that if $G$ is a group of order 165 then $G$ has no elements of order 7.
Solution.

(1) Since $\text{ord}(s) = 2$, it follows that $|H| = \text{ord}(s) = 2$. By Lagrange’s Theorem, we have
\[|D_{24} : H| = \frac{|D_{24}|}{|H|} = \frac{24}{2} = 12.\]

(2) Let $G$ be a group with $|G| = 165$. We have $165 = 55 \cdot 3 = 11 \cdot 5 \cdot 3$. Thus $7$ divides $|G|$ and therefore $G$ has no elements of order 7.

Problem 4.[10 points]
For each of the following quotient groups, find a non-quotient group isomorphic to it.

2. $C^\times / S^1$.
3. $GL(2, \mathbb{C})/SL(2, \mathbb{C})$.

Solution.

(1) We have $\epsilon : S_6 :\rightarrow \{\{1, -1\}, \}$ is a surjective homomorphism with $\ker(\epsilon) = A_9$. Therefore, by the First Isomorphism Theorem, $S_6/A_9 \cong \{\{1, -1\}, \}$.

(2) Consider the map $f: C^\times \rightarrow (\mathbb{R}_{\geq 0}, \cdot)$ given by $f(z) = |z|$ for $z \in C^\times$. Then $f$ is a surjective homomorphism with $\ker(f) = S^1$. Therefore, by the First Isomorphism Theorem, $C^\times / S^1 \cong (\mathbb{R}_{\geq 0}, \cdot)$.

(3) Consider the map $\text{det} : GL(2, \mathbb{C}) \rightarrow (\mathbb{C}^\times, \cdot)$. Then $\text{det}$ is a surjective homomorphism with $\ker(\text{det}) = SL(2, \mathbb{C})$. Therefore, by the First Isomorphism Theorem, $GL(2, \mathbb{C})/SL(2, \mathbb{C}) \cong (\mathbb{C}^\times, \cdot)$.

Problem 5.[10 points]

(1) Find all subgroups of $(\mathbb{Z}/18\mathbb{Z}, +)$ and the orders of these subgroups.

(2) List all the elements of order 18 in the group $(\mathbb{Z}/18\mathbb{Z}, +)$.

Solution.

(1) The group $\mathbb{Z}/18\mathbb{Z}$ is cyclic of order 18, and $\mathbb{Z}/18\mathbb{Z} = \langle x \rangle$, where $x = [1]_{18}$. The positive divisors of 18 are 1, 2, 3, 6, 9, 18. Therefore the subgroups of $\mathbb{Z}/18\mathbb{Z}$ are:

- $H_1 = \langle [1]_{18} \rangle = \mathbb{Z}/18\mathbb{Z}, \quad |H_1| = 18$
- $H_2 = \langle [2]_{18} \rangle = \{[0]_{18}, [2]_{18}, [4]_{18}, [6]_{18}, [8]_{18}, [10]_{18}, [12]_{18}, [14]_{18}, [16]_{18} \} \quad |H_2| = 18/2 = 9$
- $H_3 = \langle [3]_{18} \rangle = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18} \} \quad |H_3| = 18/3 = 6$
- $H_4 = \langle [6]_{18} \rangle = \{[0]_{18}, [6]_{18}, [12]_{18} \} \quad |H_4| = 18/6 \cdot 3 = 3$
- $H_5 = \langle [9]_{18} \rangle = \{[0]_{18}, [9]_{18} \} \quad |H_5| = 18/9 = 2$
- $H_6 = \langle [0]_{18} \rangle = \{[0]_{18} \} \quad |H_6| = 1.$

(2) Among integers $1 \leq m < 18$, the integers co-prime with 18 are 1, 5, 7, 11, 13. Therefore the elements of order 18 in the cyclic group $\mathbb{Z}/18\mathbb{Z}$ of order 18 are: $[1]_{18}, [5]_{18}, [7]_{18}, [11]_{18}, [13]_{18}, [17]_{18}$. 