Complex numbers.

1. Definition and notations

A complex number \( z \) is a formal expression of the form \( z = x + yi \) where \( x, y \in \mathbb{R} \). Denote by

\[ C = \{ x + yi \mid x, y \in \mathbb{R} \} \]

the set of all complex numbers. For a complex number \( z = x + yi \in C \), where \( x, y \in \mathbb{R} \) put

\[
Re(z) := x, \quad \text{the real part of } z, \\
Im(z) := y, \quad \text{the imaginary part of } z, \\
|z| := \sqrt{x^2 + y^2}, \quad \text{the absolute value of } z, \\
z := x - yi, \quad \text{the complex conjugate of } z.
\]

By convention, for \( x \in \mathbb{R} \) we identify \( z = x + 0i \in \mathbb{C} \) with \( x \in \mathbb{R} \) and write \( x + 0i = x \). Thus we have \( \mathbb{R} \subseteq \mathbb{C} \) and, moreover, \( \mathbb{R} = \{ z \in \mathbb{C} \mid Im(z) = 0 \} \).

Also by convention we write \( i = 0 + 1i \), \(-i = 0 + (-1)i \), and, more generally, \( yi = 0 + yi \) where \( y \in \mathbb{R} \). Thus \( i, -i, 2i, \sqrt{5}i \in \mathbb{C} \).

Note that for \( z \in \mathbb{C} \) we have \( |z| = 0 \) if and only if \( z = 0 \).

2. Algebraic operations on complex numbers

2.1. Addition, multiplication and complex conjugation. For \( z_1 = x_1 + y_1i, z_2 = x_2 + y_2i \in \mathbb{C} \) (where \( x_1, x_2, y_1, y_2 \in \mathbb{R} \)) define

\[
z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2) \\
z_1z_2 := x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1),
\]

so that \( z_1 + z_2 \in \mathbb{C} \) and \( z_1z_2 \in \mathbb{C} \). Note that, following the above definition of multiplication for complex numbers, we have

\[
i^2 = (0 + 1i)(0 + 1i) = (0^2 - 1^2) + i(0 \cdot 1 + 1 \cdot 0) = -1 + 0i = -1,
\]

so that \( i^2 = -1 \) in \( \mathbb{C} \).

Also, for \( z = x + iy \in \mathbb{C} \) (where \( x, y \in \mathbb{R} \)) put

\[
-z := -x + (-y)i,
\]

so that \( -z = (-1)z \).

2.2. Basic properties of addition, multiplication and complex conjugation. We have:
\[ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad \text{for any } z_1, z_2, z_3 \in \mathbb{C}, \]
\[ z_1 + z_2 = z_2 + z_1 \quad \text{for any } z_1, z_2 \in \mathbb{C}, \]
\[ z + 0 = 0 + z = z \quad \text{for any } z \in \mathbb{C}, \]
\[ z + (-z) = (-z) + z = 0 \quad \text{for any } z \in \mathbb{C}, \]
\[ z_1(z_2z_3) = (z_1z_2)z_3 \quad \text{for any } z_1, z_2, z_3 \in \mathbb{C}, \]
\[ z_1z_2 = z_2z_1 \quad \text{for any } z_1, z_2 \in \mathbb{C}, \]
\[ 1 \cdot z = z \cdot 1 = z \quad \text{for any } z \in \mathbb{C}, \]
\[ 0 \cdot z = z \cdot 0 = 0 \quad \text{for any } z \in \mathbb{C}, \]
\[ z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad \text{for any } z_1, z_2, z_3 \in \mathbb{C}, \]
\[ z\overline{z} = |z|^2 = x^2 + y^2 = |z|^2 \in \mathbb{R} \text{ for any } z = x + iy \in \mathbb{C}, \text{ where } x, y \in \mathbb{R}, \]
\[ z\overline{z} = 0 \iff z = 0 \quad \text{for any } z \in \mathbb{C}, \]
\[ \overline{z} = z \quad \text{for any } z \in \mathbb{C}, \]
\[ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{for any } z_1, z_2 \in \mathbb{C}, \]
\[ \overline{z_1z_2} = \overline{z_1}\overline{z_2} \quad \text{for any } z_1, z_2 \in \mathbb{C}, \]
\[ \overline{z} = z \iff z \in \mathbb{R} \text{ where } z \in \mathbb{C}. \]

2.3. Division of complex numbers. Recall that for any \( z \in \mathbb{C} \) we have \( z\overline{z} = |z|^2 \) and that \( z\overline{z} = |z|^2 > 0 \) whenever \( z \neq 0 \).

For \( z_1 = x_1 + y_1i, z_2 = x_2 + y_2i \in \mathbb{C} \) where \( x_1, x_2, y_1, y_2 \in \mathbb{R} \) such that \( z_2 \neq 0 \) (and hence \( |z_2| > 0 \)) we put
\[
\frac{z_1}{z_2} := \frac{1}{|z_2|^2} z_1\overline{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}i.
\]
In particular, for \( z = x + yi \neq 0 \) (where \( x, y \in \mathbb{R} \))
\[
\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2}i.
\]

2.4. Properties of division. We have:
\[
\frac{1}{z} \cdot z = z \cdot \frac{1}{z} = 1 \quad \text{for any } z \neq 0, z \in \mathbb{C},
\]
\[
\frac{1}{z} \neq 0 \quad \text{for any } z \neq 0, z \in \mathbb{C},
\]
\[
\left( \frac{z_1}{z_2} \right) = \frac{\overline{z_1}}{\overline{z_2}} \quad \text{for any } z_1, z_2 \in \mathbb{C}, z_2 \neq 0,
\]
\[
\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{for any } z_1, z_2 \in \mathbb{C}, z_2 \neq 0,
\]
\[
\left| \frac{1}{z} \right| = \frac{1}{|z|} \quad \text{for any } z \in \mathbb{C}, z \neq 0.
\]
Let $z = x + yi \in \mathbb{C}$, where $x, y \in \mathbb{R}$ and let $(r, \theta)$ be the polar coordinates of $(x, y) \in \mathbb{R}^2$, where $r \geq 0$, $0 \leq \theta < 2\pi$. In this case we also say that $(r, \theta)$ are the polar coordinates of $z$.

We have:

1. $r = |z|$, $x = r \cos \theta$, $y = r \sin \theta$ where $z = x + yi \in \mathbb{C}, x, y \in \mathbb{R}$,

2. $z = r(\cos \theta + i \sin \theta)$, where $z \in \mathbb{C}$, and $(r, \theta)$ are the polar coordinates of $z$,

3. $(r_1(\cos \theta_1 + i \sin \theta_1))(r_2(\cos \theta_2 + i \sin \theta_2)) = r_1r_2(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i)$,

   where $r_1, r_2 \geq 0$, and $\theta_1, \theta_2 \in \mathbb{R}$. Thus for the product of two complex numbers the polar radius of their product is the product of their polar radii and the polar angle of their product is the sum (modulo $2\pi$) of their polar angles.

4. For any $\theta \in \mathbb{R}$ put

   $$e^{i\theta} := 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots = \cos \theta + i \sin \theta.$$

5. Thus for any $z \in \mathbb{C}$ we have

   $$z = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

   where $(r, \theta)$ are the polar coordinates of $z$. Moreover, If

   $$z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$$

   where $r_1, r_2 \geq 0$ and $\theta_1, \theta_2 \in \mathbb{R}$ then

   $$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

6. If

   $$z = re^{i\theta},$$

   where $r > 0$ and $\theta \in \mathbb{R}$ then for every $n \in \mathbb{Z}$ we have

   $$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i(n\theta)).$$

   In particular, for $z \neq 0$, we have

   $$\frac{1}{z} = z^{-1} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos \theta - i \sin \theta).$$
Remark. The formula in part (3) above is a good way to derive the trigonometric identities for \( \cos(\theta_1 + \theta_2) \) and \( \sin(\theta_1 + \theta_2) \) if you ever forget these identities.

Indeed, let \( \theta_1, \theta_2 \in \mathbb{R} \). Then, using the definition of complex multiplication, we have

\[
(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = \\
(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) = \\
\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2),
\]

where the last equality follows from statement (3) above. Hence

\[
\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2
\]

and

\[
\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2.
\]

2.6. Roots of unity. Using the above formulas for complex multiplication in polar coordinates (specifically, part (6) above) it is easy to deduce that if \( n \geq 1 \) and

\[
\zeta_k = \cos\left(\frac{2\pi + 2\pi k}{n}\right) + i \sin\left(\frac{2\pi + 2\pi k}{n}\right) = e^{i \frac{2\pi + 2\pi k}{n}},
\]

where \( k = 0, 1, \ldots, n-1 \), then

\[
\zeta_k^n = 1.
\]

The numbers \( \zeta_k \), (where \( k = 0, 1, \ldots, n-1 \)) are called the \( n \)-th roots of unity in \( \mathbb{C} \). The number \( \zeta_0 = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right) \) is called the main \( n \)-th root of unity in \( \mathbb{C} \).

It is not hard to check that for any \( n \geq 1 \) the numbers \( \zeta_0, \ldots, \zeta_{n-1} \in \mathbb{C} \) are distinct and that they are the only solutions of the equation

\[
z^n = 1
\]

in \( \mathbb{C} \).

3. Fundamental theorem of algebra

The Fundamental Theorem of Algebra states the following:

Let \( f(z) = a_n z^n + \ldots + a_1 z + a_0 \), (where \( a_0, \ldots, a_n \in \mathbb{C} \), \( a_n \neq 0 \)) be a polynomial of degree \( n \geq 1 \) with coefficients in \( \mathbb{C} \). (Here \( z \) is a variable.) Then there exist complex numbers \( z_1, \ldots, z_n \in \mathbb{C} \) such that \( f(z) \) factor as

\[
f(z) = a_n (z - z_1) \ldots (z - z_n).
\]

Moreover, one can show that in the above situation for \( z_* \in \mathbb{C} \) we have \( f(z_*) = 0 \) if and only if \( z_* = z_k \) for some \( 1 \leq k \leq n \), that is \( z_1, \ldots, z_n \) are the only solutions of the equation \( f(z) = 0 \) in \( \mathbb{C} \).
In particular, if a polynomial \( f(z) = a_n z^n + \ldots + a_1 z + a_0 \) of degree \( n \) as above has \( n \) distinct roots \( z_1, \ldots, z_n \) in \( \mathbb{C} \), then
\[
f(z) = a_n (z - z_1) \ldots (z - z_n).
\]

Applying this fact to the equation \( z^n - 1 = 0 \) (where \( n \geq 1 \)) we conclude that for every \( n \geq 1 \) the polynomial \( z^n - 1 \) factors as
\[
z^n - 1 = (z - \zeta_0) \ldots (z - \zeta_{n-1})
\]
where \( \zeta_0, \ldots, \zeta_{n-1} \) are the \( n \)-th roots of unity in \( \mathbb{C} \).

### 3.1. Factorization of polynomials with real coefficients

Let
\[
f(z) = a_n z^n + \ldots + a_1 z + a_0
\]
be a polynomial of degree \( n \geq 1 \) with real coefficients \( a_k \in \mathbb{R} \), for \( k = 0, \ldots, n \), and \( a_n \neq 0 \).

Suppose that \( z_s = x + iy \in \mathbb{C} \) is a root of the equation \( f(z) = 0 \), so that \( f(z_s) = 0 \). Then
\[
0 = f(z_s) = a_n z_s^n + \ldots + a_1 z_s + a_0 = a_n (z_s)^n + \ldots + a_1 z_s + a_0 = \left( a_n (\overline{z_s})^n + \ldots + a_1 \overline{z_s} + a_0 \right)
\]
since all \( a_k \in \mathbb{R} \)
\[
= a_n (\overline{z_s})^n + \ldots + a_1 \overline{z_s} + a_0 = f(\overline{z_s})
\]
Thus \( \overline{z_s} = x - iy \) is also a root of \( f(z) = 0 \). Note that \( \overline{z_s} = z_s \) if and only if \( z_s \in \mathbb{R} \).

Hence we see that those complex roots of \( f(z) = 0 \) that do not belong to \( \mathbb{R} \), come in pairs of the form \( z_s, \overline{z_s} \).

Suppose that \( z_s = x + iy \in \mathbb{C} \) is a root of \( f(z) = 0 \) where \( x, y \in \mathbb{R} \) and \( z_s \notin \mathbb{R} \), that is \( y \neq 0 \). Then
\[
(z - z_s)(z - \overline{z_s}) = (z - x - iy)(z - x + iy) = (z - x)^2 + y^2 = z^2 - 2xz + x^2 + y^2.
\]
Since \( x, y \in \mathbb{R} \), the polynomial (in the variable \( z \)) \( z^2 - 2xz + x^2 + y^2 \) has coefficients in \( \mathbb{R} \). Moreover, the quadratic formula and the assumption that \( y \neq 0 \) imply that the equation \( z^2 - 2xz + x^2 + y^2 = 0 \) in the variable \( z \) has no solutions in \( \mathbb{R} \).

Now let
\[
f(z) = a_n (z - z_1) \ldots (z - z_n)
\]
be the factorization of \( f(z) \) provided by the Fundamental Theorem of Algebra. If \( z_k \in \mathbb{R} \) is a real number, the term \( z - z_k \) is a polynomial of degree 1 in \( z \) with real coefficients. The roots of \( f(z) \) which are not real numbers, come in conjugate pairs the form \( z_s, \overline{z_s} \). With a small extra argument we can show that \( z_s \) and \( \overline{z_s} \) have the same multiplicities in the above factorization of \( f(z) \).

Hence it follows from (†) above that the polynomial \( f(z) \) can be written as a product of polynomials of degree 1 and 2 with real coefficients.
This fact about factorization of polynomials $f(z)$ with coefficients in $\mathbb{R}$ as a product of linear and quadratic terms (also with coefficients in $\mathbb{R}$) plays a key role in the theory of partial fractions (which you must have seen in Calculus II) and in solving linear differential equations with constant coefficients.