Homework 11

Problem 1 Let \( R \) be a countable ring. Prove that the following conditions are equivalent:

1. The ring \( R \) is noetherian.
2. There does not exist a strictly ascending infinite chain of ideals in \( R \):
   \[ I_1 < I_2 < \cdots < I_n < \cdots \]

Problem 2 Let \( R = \mathbb{Z}[x_1, x_2, \ldots] \). Let \( I = (x_1, x_2, \ldots) \triangleleft R \). Prove that \( I \) is not finitely generated.

Problem 3 Let \( S \) be a nonempty subset of \( \mathbb{C}^n \).

Prove that there exist \( f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n] \) such that \( f_i|_S = 0 \) for \( 1 \leq i \leq k \) and such that for every \( g \in \mathbb{C}[x_1, \ldots, x_n] \) satisfying \( g|_S = 0 \) there exist \( g_1, \ldots, g_k \in \mathbb{C}[x_1, \ldots, x_n] \) such that
\[ g = f_1g_1 + \cdots + f_kg_k. \]

Problem 4 Prove that if \( R \) is a noetherian ring and \( I \triangleleft R \) is an ideal in \( R \) then the ring \( R/I \) is also noetherian.

Problem 5 Prove that if \( R, S \) are noetherian ring then the ring \( R \times S \) is also noetherian.

Problem 6 Let \( F \) be a field. An \( F \)-algebra is a vector space \( R \) over \( F \) such that \( R \) is also a ring and such that
\[ (\alpha a)b = \alpha(ab) = a(\alpha b) \] for all \( \alpha \in F, a, b \in R \).

Let \( R \) be an \( F \)-algebra which is finite-dimensional as a vector-space over \( F \). Prove that \( R \) is a noetherian ring.

Problem 7∗∗[optional] Prove that if \( R \) is a noetherian ring then the ring of formal power series \( R[[x]] \) is also noetherian.