Differential forms

1. Differential forms at a point

1.1. Definition and examples of k-forms.

Definition 1.1. Let $p \in \mathbb{R}^n$ and let $k \geq 1$ be an integer.

A k-form at $p$ on $\mathbb{R}^n$ is a function

$$\omega : \mathbb{R}^n_p \times \ldots \mathbb{R}^n_p \rightarrow \mathbb{R}$$

such that $\omega$ satisfies the following properties:

1. The map $\omega$ is multi-linear, that is we have

$$\omega(\bar{a}_1, \ldots, c\bar{a}_i + c'\bar{a}'_i, \ldots \bar{a}_k) = c\omega(\bar{a}_1, \ldots, \bar{a}_i, \ldots \bar{a}_k) + c'\omega(\bar{a}_1, \ldots, \bar{a}'_i, \ldots \bar{a}_k)$$

for every $1 \leq i \leq k$, for all $\bar{a}_1, \ldots, \bar{a}_i, \bar{a}'_i, \ldots \bar{a}_k \in \mathbb{R}^n_p$ and all $c, c' \in \mathbb{R}$.

2. The map $\omega$ is alternating or anti-symmetric, that is,

$$\omega(\bar{a}_1, \ldots, \bar{a}_i, \ldots, \bar{a}_j, \bar{a}_k) = -\omega(\bar{a}_1, \ldots, \bar{a}_j, \ldots, \bar{a}_i, \bar{a}_k)$$

for all $1 \leq i < j \leq k$ and all $\bar{a}_1, \ldots, \bar{a}_k \in \mathbb{R}^n_p$.

Also, by convention, for $k = 0$, we say a 0-form at $p$ is any real number $r \in \mathbb{R}$.

For $k \geq 0$ we denote the set of all $k$-forms at $p$ on $\mathbb{R}^n$ by $\Omega^k_p \mathbb{R}^n$.

Remark 1.2. For $k = 1$ condition (1) in Definition 1.1 just means that a 1-form at $p$ is a linear map $\alpha : \mathbb{R}^n_p \rightarrow \mathbb{R}$; in this case condition (1) in Definition 1.1 is vacuously satisfied. Thus for $k = 1$ Definition 1.1 agrees with the definition of a 1-form given in Ch 1.5 of O'Neill.

Remark 1.3. For $k \geq 0$ the set $\Omega^k_p \mathbb{R}^n$ has a natural structure of a vector space over $\mathbb{R}$, with respect to the pointwise addition and pointwise multiplication by a scalar of $k$-forms as functions $(\mathbb{R}^n_p)^k \rightarrow \mathbb{R}$.

Example 1.4.

1. Consider the standard dot-product on $\mathbb{R}^3_p$:

$$\omega : \mathbb{R}^3_p \times \mathbb{R}^3_p \rightarrow \mathbb{R}$$

defined as

$$\omega((v_1, v_2, v_3)_p, (w_1, w_2, w_3)_p) = v_1w_1 + v_2w_2 + v_3w_3$$

Then $\omega$ satisfies condition (1) of Definition 1.1 but does not satisfy condition (2). Thus $\omega$ is not a 2-form at $p$.

2. For a point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ consider a map $\omega : \mathbb{R}^3_p \times \mathbb{R}^3_p \rightarrow \mathbb{R}$ defined as

$$\omega((v_1, v_2, v_3)_p, (w_1, w_2, w_3)_p) = e^{p_1}v_1w_3 - e^{p_1}v_3w_1.$$ 

This map is a 2-form at $p$. 

1
(3) For \( p \in \mathbb{R}^n \) consider the map
\[
\omega : \mathbb{R}^n_p \times \ldots \times \mathbb{R}^n_p \rightarrow \mathbb{R}
\]
defined as
\[
\omega(\vec{a}_1, \ldots, \vec{a}_n) := \det[\vec{a}_1 | \ldots | \vec{a}_n]
\]
where \( \vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^3_p \) and in the above matrix we write \( \vec{a}_i \) as column-vectors.
Then \( \omega \) is an \( n \)-form at \( p \).

(4) For a point \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \) consider a map \( \omega : \mathbb{R}^3_p \times \mathbb{R}^3_p \rightarrow \mathbb{R} \) defined as
\[
\omega((v_1, v_2, v_3)_p, (w_1, w_2, w_3)_p) = v_1^2w_3^2 - v_2^2w_1^2.
\]
This map satisfies condition (2) of Definition 1.1 but does not satisfy condition (1) of Definition 1.1. Thus \( \omega \) is not a 2-form at \( p \).

Condition (2) of Definition 1.1 easily implies:

**Lemma 1.5.** Let \( k \geq 2 \) and \( \omega \in \Omega^k_p \mathbb{R}^n \). Then
\[
\omega(\vec{a}_1, \ldots, \vec{a}_i, \ldots, \vec{a}_j, \ldots, \vec{a}_k) = 0
\]
whenever \( 1 \leq i < j \leq k \) and \( \vec{a}_1, \ldots, \vec{a}_k \in \mathbb{R}^n_p \) are such that \( \vec{a}_i = \vec{a}_j \).

1.2. Coefficients and basic properties of \( k \)-forms.

**Definition 1.6** (Coefficients of a \( k \)-form). Let \( \omega \in \Omega^k_p \mathbb{R}^n \), where \( k \geq 1 \). For \( 1 \leq i_1, \ldots, i_k \leq n \) put
\[
\omega_{i_1, \ldots, i_k} := \omega((e_{i_1})_p, \ldots, (e_{i_k})_p).
\]
We call the numbers \( \omega_{i_1, \ldots, i_k} \) the coefficients of \( \omega \).

We will often use a shorthand notation \( \vec{I} = (i_1, \ldots, i_k) \) and \( \omega_{\vec{I}} := \omega_{i_1, \ldots, i_k} \).

The fact that a \( k \)-form is a multi-linear map implies that a \( k \)-form is uniquely determined by its coefficients:

**Proposition 1.7.** Let \( \omega, \omega' \in \Omega^k_p \mathbb{R}^n \), where \( k \geq 1 \), be such that
\[
\omega_{i_1, \ldots, i_k} = \omega'_{i_1, \ldots, i_k}
\]
for all \( k \)-tuples of indices \( 1 \leq i_1, \ldots, i_k \leq n \).
Then \( \omega = \omega' \) (that is, for every \( \vec{a}_1, \ldots, \vec{a}_k \in \mathbb{R}^n_p \) we have \( \omega(\vec{a}_1, \ldots, \vec{a}_k) = \omega'(\vec{a}_1, \ldots, \vec{a}_k) \)).

**Lemma 1.8.** Let \( k \geq 1 \), \( p \in \mathbb{R}^n \) and let \( \omega \in \Omega^k_p \mathbb{R}^n \). Then:

1. If \( 1 \leq i_1, \ldots, i_k \leq n \) are such that for some \( t \neq s \) \( i_t = i_s \) then \( \omega_{i_1, \ldots, i_k} = 0 \).
2. If \( k \geq n + 1 \) then for any \( 1 \leq i_1, \ldots, i_k \leq n \) we have \( \omega_{i_1, \ldots, i_k} = 0 \) and consequently \( \omega = 0 \). Thus for \( k \geq n + 1 \) \( \Omega^k_p \mathbb{R}^n = \{0\} \).
(3) Let \( \vec{I} = (i_1, \ldots, i_k) \), \( \vec{J} = (j_1, \ldots, j_k) \) be \( k \)-tuples of distinct indices from \( \{1, \ldots, n\} \), such that \( \vec{J} \) is a re-arrangement of \( \vec{I} \).

Then \( \omega_{\vec{J}} = \omega_{\vec{I}} \) if \( \vec{J} \) can be obtained from \( \vec{I} \) by an even number of “swaps” (where a “swap” in a \( k \)-tuple of indices consists of interchanging two entries in this \( k \)-tuple while leaving the other entries fixed), and \( \omega_{\vec{J}} = -\omega_{\vec{I}} \) if \( \vec{J} \) can be obtained from \( \vec{I} \) by an odd number of “swaps”.

(4) For \( 1 \leq k \leq n \) a \( k \)-form \( \omega \) at \( p \) is uniquely determined by its coefficients \( \omega_{i_1, \ldots, i_k} \) corresponding to the strictly increasing \( k \)-tuples of indices \( 1 \leq i_1 < \cdots < i_k \). Thus is, if \( 1 \leq k \leq n \) and \( \omega, \omega' \in \Omega^k_p \mathbb{R}^n \) are such that for every \( 1 \leq i_1 < \cdots < i_k \) we have \( \omega_{i_1, \ldots, i_k} = \omega'_{i_1, \ldots, i_k} \) then \( \omega = \omega' \).

**Notation 1.9.** Let \( \vec{I} = (i_1, \ldots, i_k) \) and \( \vec{J} = (j_1, \ldots, j_k) \) be where \( 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_k \leq n \).

We denote

\[
\delta_{\vec{J}}^\vec{I} :=
\begin{cases}
1 & \text{if } j_1, \ldots, j_k \text{ are distinct and } \vec{J} \text{ can be obtained from } \vec{I} \text{ by an even number of swaps} \\
-1 & \text{if } j_1, \ldots, j_k \text{ are distinct and } \vec{J} \text{ can be obtained from } \vec{I} \text{ by an odd number of swaps} \\
0 & \text{otherwise}
\end{cases}
\]

Note that we always have \( \delta_{\vec{J}}^\vec{I} = \delta_{\vec{J}}^\vec{I} \).

1.3. Basic \( k \)-forms.

**Proposition-Definition 1.10** (Basic \( k \)-forms). Let \( p \in \mathbb{R}^n \), \( 1 \leq k \leq n \) and let \( 1 \leq i_1, \ldots, i_k \leq n \).

Then there exists a unique \( k \)-form \( \omega \) at \( p \), denoted \( dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p \) such that for all \( 1 \leq j_1, \ldots, j_k \leq n \) we have

\[
\omega_{j_1, \ldots, j_k} = \delta_{i_1, \ldots, i_k}^{j_1, \ldots, j_k}.
\]

For \( \vec{I} = (i_1, \ldots, i_k) \) we will also denote \( dx_{\vec{I}}|_p := dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p \).

Thus

\[
dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p(e_{j_1})_p, \ldots, (e_{j_k})_p = \delta_{i_1, \ldots, i_k}^{j_1, \ldots, j_k}.
\]

**Remark 1.11.** Proposition-Definition 1.10 implies that:

(a) If \( 1 \leq i_1, \ldots, i_k \leq n \) and \( i_r = i_s \) for some \( r \neq s \) then \( dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p = 0 \).

(b) \( \vec{I} = (i_1, \ldots, i_k) \), \( \vec{J} = (j_1, \ldots, j_k) \) and if \( \vec{J} \) is a rearrangement of \( \vec{I} \) then either \( dx_{\vec{J}}|_p = dx_{\vec{I}}|_p \) or \( dx_{\vec{J}}|_p = -dx_{\vec{I}}|_p \).

**Example 1.12.** Let \( \omega \) be the “determinant” \( n \)-form at \( p \in \mathbb{R}^n \) from part (3) of Example 1.4 above. Then \( \omega = dx_1|_p \wedge \cdots \wedge dx_n|_p \).

Indeed, by Proposition 1.7 to verify this equality we just need to check that the \( n \)-forms \( \omega dx_1|_p \wedge \cdots \wedge dx_n|_p \) have the same coefficients for all \( 1 \leq
corresponding to \( i \). Let Theorem 1.13.

Theorem 1.13. Let \( 1 \leq k \leq n \) and \( p \in \mathbb{R}^n \). Then:

1. The space \( \Omega^k_p \mathbb{R}^n \) is a finite-dimensional vector space, with a basis given by
   \[
   \{ dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p | 1 \leq i_1 < \cdots < i_k \leq n \}.
   \]
2. We have \( \dim \Omega^k_p \mathbb{R}^n = \binom{n}{k} \).
3. For any \( \omega \in \Omega^k_p \mathbb{R}^n \) we have
   \[
   \omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1, \ldots, i_k} dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega((e_{i_1})_p, \ldots, (e_{i_k})_p) dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p
   \]

1.4. Wedge-product.

Proposition-Definition 1.14. Let \( r, s \geq 1 \) and let \( \alpha \in \Omega^r_p \mathbb{R}^n \) and \( \beta \in \Omega^s_p \mathbb{R}^n \).

Define a function
\[
\alpha \wedge \beta : (\mathbb{R}^n)_p^{r+s} \to \mathbb{R}
\]
as
\[
(\alpha \wedge \beta)(\vec{a}_1, \ldots, \vec{a}_{r+s}) := \sum_{1 \leq i_1, \ldots, i_{r+s} \leq r+s} \delta^{i_1 \cdots i_{r+s}}_{1 \cdots r+s} \alpha(\vec{a}_{i_1}, \ldots, a_{i_s}) \beta(\vec{a}_{i_{r+1}}, \vec{a}_{i+s})
\]
where the summation is taken over all rearrangements \( i_1, \ldots, i_{r+s} \) of \( 1, 2, \ldots, r+s \).

Then \( \alpha \wedge \beta \) is an \((r+s)\) form at \( p \), called the wedge-product of \( \alpha \) and \( \beta \).

Also, if \( r = 0 \) and \( \alpha \) is a 0-form at \( p \), that is \( \alpha = c \) for some number \( c \in \mathbb{R} \), for any \( s \)-form \( \beta \in \Omega^s_p \mathbb{R}^n \) (where \( s \geq 0 \)) we put \( \alpha \wedge \beta := c \beta \).

Similarly, if \( s = 0 \) and \( \alpha \) is a 0-form at \( p \), that is \( \alpha = c \) for some number \( c \in \mathbb{R} \), for any \( r \)-form \( \beta \in \Omega^r_p \mathbb{R}^n \) (where \( r \geq 0 \)) we put \( \beta \wedge \alpha = \alpha \wedge \beta := c \beta \).
**Proposition 1.15.** The $\wedge$ operation on forms at $p \in \mathbb{R}^n$ satisfies the following properties:

1. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$;
2. $(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta$;
3. $(c \alpha) \wedge \beta = c(\alpha \wedge \beta)$ (where $c \in \mathbb{R}$);
4. if $\alpha \in \Omega^r_p \mathbb{R}^n$ and $\beta \in \Omega^s_p \mathbb{R}^n$ (where $r \geq 0, s \geq 0$) then
   \[ \alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha; \]
5. $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

**Remark 1.16.** Consider a form $\omega = dx_{i_1}|_p \wedge \cdots \wedge dx_{i_r}|_p$, as defined in Proposition-Definition 1.10 above. Note that $dx_{i_1}|_p, \ldots, dx_{i_r}|_p \in \Omega^1_p \mathbb{R}^n$.

Then

\[ \omega = dx_{i_1}|_p \wedge \cdots \wedge dx_{i_r}|_p \]

where the right-hand side is interpreted as iteratively applying the wedge-product operation from Proposition-Definition 1.14.